

Reflecting magnons

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Abstract

We study the worldsheet reflection matrix of a string attached to a D-brane in $AdS_5 \times S^5$. The D-brane corresponds to a maximal giant graviton and it wraps an S^3 inside S^5 . In the gauge theory, the open string is described by a spin chain with boundaries. We study an open string with a large $SO(6)$ charge, which allows us to focus on one boundary at a time and to define an asymptotic boundary reflection matrix. We consider two cases corresponding to two possible relative orientations for the charges of the giant graviton and the open string. Using the symmetries of the problem we compute the boundary reflection matrix up to a phase. These matrices obey the boundary Yang Baxter equation. A crossing equation is derived for the overall phase. We perform weak coupling computations up to two loops and obtain results that are consistent with integrability. Finally, we determine the phase factor at strong coupling using classical solutions.

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1 Introduction

Recently there has been a great deal of progress in understanding planar $\mathcal{N} = 4$ super Yang Mills, see [1, 2, 3, 4, 5, 6, 7] and references therein. Planar Yang Mills theories give rise to a two dimensional theory which can be viewed as the worldsheet of a string. From the gauge theory point of view, single trace operators give rise to a closed spin chain, which in turn is related to a two dimensional field theory on a circle. When the charges of the state under consideration are very large one can view the gauge fixed closed string theory [8] as living on a large circle. The limit where the string is infinite is particularly simple [9, 10] and one can solve exactly this problem [1, 2, 3, 11]. By “solving” we mean finding the fundamental excitations, their dispersion relation, and their scattering amplitudes on the infinite string for all values of the 't Hooft coupling. It is very useful to consider the symmetries of the problem, which are larger than naively expected [1]. These symmetries determine completely the matrix structure of the two particle scattering matrix [1, 12]. The remaining phase can then be determined by using a crossing symmetry equation [2, 3].

In integrable field theories it is often possible to define the system on a half line, with suitable boundary conditions such that the system remains integrable. A nice example is the boundary Sine-Gordon theory studied in [13]. In this article we study some physical problems in $\mathcal{N} = 4$ super Yang Mills that lead to a system with a boundary. From the string theory point of view we expect to have boundaries when we have D-branes. Then the open string excitations are described by a two dimensional field theory with a boundary. Such D-branes can arise in several situations:

- Gauge theories with additional flavors. Open strings correspond to strings with a quark and an anti-quark at the ends.
- Theories with lower dimensional defects, which in some cases can be realized as D-branes in the bulk [14].
- Certain large charge operators in $\mathcal{N} = 4$ super Yang mills. For example, operators of charge N of the form $\det(Z)$, where Z is one of the complex scalar fields in the theory. We will focus on such operators and their excitations in this paper [15, 16].

Another case where integrable systems with boundaries arise is when we consider operator insertions along a Wilson loop [17]. This is a situation where, despite the absence

of explicit D-branes in the bulk, we end up with a system with a boundary. Of course, we could say that a Wilson line is an open string which ends on the boundary of AdS_5 .

Previous work analyzing open spin chains in $\mathcal{N} = 4$ super Yang Mills or the corresponding open strings with various boundary conditions includes [14, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. We focus, mainly, on two intimately related cases which consist of giant graviton operators with two possible orientations relative to the open string ground state. We show that in one case we have boundary degrees of freedom, while in the other case we do not.

The central idea in this paper is a generalization of the analysis by Beisert [1, 12] to the case where we have boundaries. Namely, we will use the symmetries of the system to determine the matrix structure of the boundary scattering matrix. We then proceed to write a crossing equation for the phase factor. Although we have not solved the crossing equation, we have computed the phase factor at weak and strong coupling.

We have also checked that the boundary Yang Baxter equation is obeyed. This follows by an argument similar to the one used in [12]. Furthermore, we performed calculations at two loops in the weak coupling expansion and obtained results compatible with integrability. At strong coupling, this system leads to a classically integrable boundary condition for the string sigma model [28].

When studying the action of the symmetries, it has proven to be useful to have in mind the physical picture for the extra central charges suggested by the classical string theory analysis in [34] (see also [12, 35] for a related picture). Although we explicitly discuss the specific case of giant gravitons, our methods can be extended without too much work to the various cases listed above.

This article is organized as follows. In section two we discuss the boundaries related to giant gravitons, both in string theory and in the gauge theory. In section three we derive the exact reflection matrices up to an overall phase. In section four we study these theories perturbatively in the weak coupling limit. We obtain the form of the phase factors up to two loops and we also perform some explicit checks of the exact results. In section five we carry out an analogous discussion of the strong coupling regime. We conclude with a discussion of these results in section six. Finally, we include two appendices. In appendix A we discuss the two loop integrability of the system with a boundary, while in appendix B we present explicit calculations of wave functions and reflection matrices at weak coupling.

2 Giant gravitons, determinants and boundaries

We study open strings attached to maximal giant gravitons [15] in $AdS_5 \times S^5$. These were previously studied at weak coupling at one loop in [23] and at two loops in [31], while a strong coupling classical analysis was carried out recently in [28]. Problems with the integrability of the theory at two loops were pointed out in [31]. We will see, however, that a non trivial extra term coming from a subtle interaction with the boundary will render the theory integrable.

2.1 Giant magnons meet giant gravitons

2.1.1 Giant gravitons

Giant gravitons are D3 branes in $AdS_5 \times S^5$ [15]. These D3 branes wrap topologically trivial cycles, but are prevented from collapsing by their coupling to the background fields. We will concentrate on the so called “maximal giant gravitons” which are D3 branes wrapping a maximum size S^3 inside S^5 . We can introduce coordinates for the S^5 in terms of $W = \Phi^1 + i\Phi^2$, $Y = \Phi^3 + i\Phi^4$ and $Z = \Phi^5 + i\Phi^6$, with $|Z|^2 + |W|^2 + |Y|^2 = 1$. Maximal giant gravitons are given by a pair of independent linear equations $a^I \Phi^I = b^I \Phi^I = 0$, and are all equivalent up to an $SO(6)$ rotation of the sphere. These configurations preserve half of the supercharges. The particular half that they preserve depends on their orientation inside the S^5 .

We are interested in studying open string excitations on the giant gravitons. Our methods work best when the open string carries a large amount of charge. Thus, we also want to single out a special generator, $J = J_{56}$, of $SO(6)$ which generates rotations in the 56 plane. We consider open strings with large charge J . In the field theory such states will involve a large number of insertions of the field Z . Since we are breaking the $SO(6)$ symmetry by selecting a particular generator, J , we find that the explicit open string description depends on the orientation of the giant graviton inside S^5 .

We will consider two cases where the D3 brane wraps the following three spheres

- The three sphere given by $Z = 0$. We will call this the $Z = 0$ giant graviton brane. We choose its orientation so that it preserves the same supersymmetries as the field Z in the field theory.

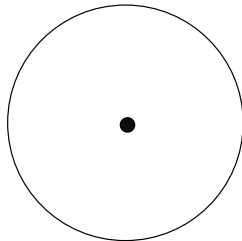


Figure 1: $Z = 0$ brane in the Z plane.

- The three sphere given by $Y = 0$, which we call the $Y = 0$ giant graviton brane. This brane preserves half of the supersymmetries preserved by the field Z in the field theory.

2.1.2 Giant magnons hitting giant gravitons

In what follows we will study open strings with a large amount of charge J . The centrifugal force pushes most of this string to the circle at $|Z| = 1$. We choose a light cone gauge so that a pointlike string moving along this great circle corresponds to the BMN vacuum [9]. In light-cone gauge the string has length J . The ground state of this string preserves half of the spacetime supersymmetries. In particular, it preserves those supercharges with $\Delta - J = 0$, where Δ is the conformal dimension. Furthermore, we can have excitations with momentum p that move along the string. The lowest energy excitation with a given momentum is BPS. It corresponds to an elementary magnon on the corresponding gauge theory spin chain. The state manages to be BPS due to the existence of additional central charges [1]. A convenient picture for the origin of these central charges is the following [34]. We draw the projection of the configuration on the Z plane. This plane is embedded in $AdS_5 \times S^5$ as explained in detail in [36]. The string ground state corresponds to a point on the rim of the circle. An elementary excitation corresponds to a segment that joins two points on the rim. The two central charges correspond to string winding charges along this Z plane [34]. It is now convenient to think about the two branes mentioned above in these coordinates. The $Z = 0$ giant graviton brane is simply a point at $Z = 0$, and it wraps an S^3 inside the S^5 , see figure 1. The $Y = 0$ giant graviton brane, on the other hand, covers the whole disk, see figure 2. At each point of the disk it also wraps an S^1 inside the S^3 that sits at that point. This circle shrinks at the rim of the disk so that we end up with a brane with the S^3 topology.

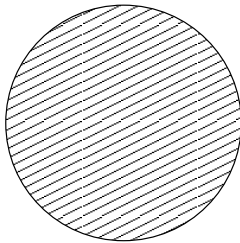


Figure 2: $Y = 0$ brane in the Z plane.

In the large J limit the string worldsheet is a very long segment, so that when we analyze the effects near one of the boundaries we can forget about the existence of the other boundary and consider the system on a half infinite line. Therefore, we consider first the problem of a giant magnon coming from infinity and bouncing off the boundary back to infinity. In particular, this means that our states interpolate between the usual vacuum of BMN states [9] and the boundary. Furthermore, this implies that one of the ends of the string looks like a “heavy” particle - i.e., there is an infinite amount of J charge at this point - moving at the speed of light in a maximum circle of S^5 , see figure 3 and [34].

Let us now look at the shape of the corresponding strings on the Z plane. The shape of this string could be complicated at a random point in worldsheet time, but in the asymptotic region (worldsheet time $t \rightarrow \pm\infty$) they must look like giant magnons. This means they connect two points on the rim of the disk. This yields no surprise for the $Y = 0$ brane: the asymptotic scattering states for the $Y = 0$ brane are just strings stretched between points on the rim. This might give the impression that the strings are contained within the D-brane. This is not necessarily true; there is an additional $S^3 \subset S^5$ at each point on the disk and the brane and the string could be separated within this S^3 .

The $Z = 0$ brane presents an interesting characteristic. In order for the string to interpolate between the correct states we are led to the following picture of the asymptotic scattering configuration, see figure 3 (b). We need to have a string that connects the rim of the disk to the center where the $Z = 0$ giant graviton brane sits.

This, in turn, suggests that the $Z = 0$ brane carries a boundary degree of freedom. Even when there is not asymptotic excitation we should have the piece of string connecting the rim of the disk to $Z = 0$, see figure 3 (c).

A string lying along a segment in the Z plane carries non vanishing central charges of

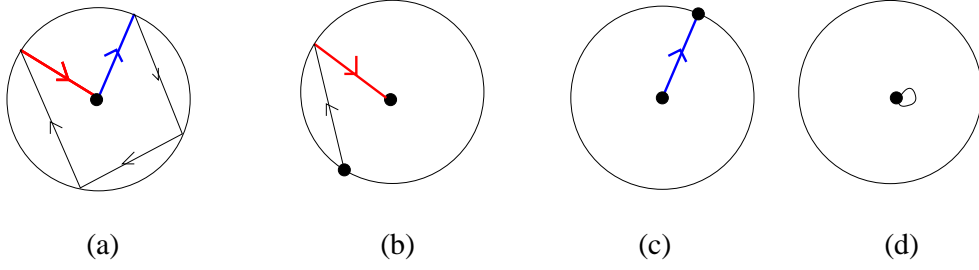


Figure 3: (a) Large J open string attached to a $Z = 0$ giant graviton brane. (b) Asymptotic form of the initial condition for the worldsheet scattering of a magnon off the right boundary. The dot on the boundary represents an infinite string in the lightcone ground state. (c) The boundary degree of freedom corresponds to a string going from the brane to the rim of the circle. (d) A string configuration for sufficiently small J does not get close to the boundary of the circle.

the worldsheet algebra, since we argued that those central charges correspond to string winding charges on the Z plane.

An important comment at this point is that strings with finite J charge never reach the asymptotic vacuum described above and consequently cannot reach the rim of the Z plane. These strings are localized around the brane at the center of the circle.

From the picture presented so far, we are lead to a simple guess for the energy of the boundary state, once we understand the representation of $SU(2|2)^2$ to which it belongs. Let us assume that it belongs to the smallest BPS representation. We will later substantiate this statement by a weak coupling computation where we check that this is indeed the case. Once this is shown for weak coupling, it will be true at all values of the coupling. This implies that the energy is $\epsilon = \sqrt{1 + |\vec{k}|^2}$ where \vec{k} are the two the central charges. We then notice that the central charge is precisely half the central charge of a magnon with momentum $p = \pi$, which corresponds to a string joining antipodal points on the rim. Therefore,

$$\epsilon_B = \sqrt{1 + 4g^2}, \quad g^2 = \frac{\lambda}{16\pi^2} \quad (2.1)$$

where λ is the 't Hooft coupling. Moreover, since the string in figure 3 (c) is sitting at a point in the $S^3 \subset S^5$ we have collective coordinates and their quantization is expected to lead to BPS boundary bound states with higher $SU(2|2)^2$ charges, as we have in the bulk [37, 38]. These states have energy $\epsilon_B(n) = \sqrt{n^2 + 4g^2}$.

These statements do not rely on integrability, only on the symmetries of the theory. Our exact and perturbative calculations presented in the following sections agree precisely with the results discussed above.

2.2 Determinants in the gauge theory: the weak coupling description

The coordinates chosen in the previous section make it easy to translate this analysis to the gauge theory side of the story. Here we think of W, Y, Z as the three complex scalars of $\mathcal{N} = 4$ super Yang Mills (and of course we also have their complex conjugates).

Then the $Z = 0$ giant graviton brane, which is the maximal giant graviton given by the equation $Z = 0$, corresponds to the gauge theory operator $\det(Z)$ [16, 39, 40, 41]. This is a gauge invariant operator with $J = N$. Of course, the $Y = 0$ giant graviton brane is then obtained by an $SO(6)$ rotation as the operator $\det(Y)$. Both of these operators correspond to the maximal giant gravitons on their ground state. We now want to consider giant gravitons with open strings attached. These are given by replacing one of the entries of the determinant by a chain similar to the one appearing in single trace operators [23, 24, 40, 42, 43, 44, 45]. For example, for the $Y = 0$ giant graviton brane we can write

$$\mathcal{O}_Y = \epsilon_{i_1 i_2 \dots i_{N-1} B}^{j_1 j_2 \dots j_{N-1} A} Y_{j_1}^{i_1} Y_{j_2}^{i_2} \dots Y_{j_{N-1}}^{i_{N-1}} (ZZZ \dots ZZZ)_A^B \quad (2.2)$$

where one can make impurities propagate inside the chain of Z s. Thus we consider operators of the form

$$\mathcal{O}_Y(\chi) = \epsilon_{i_1 i_2 \dots i_{N-1} B}^{j_1 j_2 \dots j_{N-1} A} Y_{j_1}^{i_1} Y_{j_2}^{i_2} \dots Y_{j_{N-1}}^{i_{N-1}} (\dots ZZZ \chi ZZZ \dots)_A^B \quad (2.3)$$

where χ denotes a generic impurity. For the $Z = 0$ giant graviton brane, an operator of the form (2.2) with Y replaced by Z would factorize into a determinant and a single trace [24]. This would not describe an open string but a D-brane plus a closed string. Instead we consider excitations of the form

$$\mathcal{O}_Z(\chi, \chi', \chi'') = \epsilon_{i_1 i_2 \dots i_{N-1} B}^{j_1 j_2 \dots j_{N-1} A} Z_{j_1}^{i_1} Z_{j_2}^{i_2} \dots Z_{j_{N-1}}^{i_{N-1}} (\chi ZZ \dots ZZZ \chi' ZZZ \dots ZZ \chi'')_A^B \quad (2.4)$$

where the impurities χ and χ'' are stuck at the ends of the Z -string. The impurities will reflect when they get to the ends of the string of Z s. Of course, in the large J limit, we only have to worry about one of the ends at a time.

As we mentioned above the two kinds of giant gravitons are related by an $SO(6)$ transformation. Thus, if we start with the $Z = 0$ brane and we add Y impurities so as to completely “fill” the chain we would end up with a state of the form

$$\mathcal{O}' = \epsilon_{i_1 i_2 \dots i_{N-1} B}^{j_1 j_2 \dots j_{N-1} A} Z_{j_1}^{i_1} Z_{j_2}^{i_2} \dots Z_{j_{N-1}}^{i_{N-1}} (YYY \dots YYY)_A^B \quad (2.5)$$

which is simply an $SO(6)$ transform of the state \mathcal{O} in (2.2).

3 Exact Results for the boundary reflection matrix

Following the work of Beisert [1, 12], it is possible to calculate, up to an overall phase, the reflection matrix associated with the scattering of impurities from the boundaries discussed in the previous section. All we need are the symmetries of the theory and the representations of the states involved. In order to carry out this analysis it is important to understand well the symmetries of the system. Let us first discuss the symmetries of the bulk, before we add the boundaries. As explained in [1, 12] we have a centrally extended $SU(2|2)^2$ algebra. We can consider one of these factors at a time. Each factor has eight supercharges \mathfrak{Q}_a^α and \mathfrak{S}_α^a which transform under $SU(2) \times SU(2) \subset SU(2|2)$. We denote the generators of $SU(2) \times SU(2)$ as $\mathfrak{R}_b^a, \mathcal{L}_\beta^\alpha$ respectively. We follow the notation of [1]. The algebra contains a generator $\mathfrak{E} = \frac{\epsilon}{2}$, where ϵ is the energy of an excitation around the vacuum built with Z s, $\epsilon = \Delta - J_{56}$. In addition we have two extra bosonic generators k and \bar{k} which are the extra central charges which appear in the anti-commutators¹

$$\{\mathfrak{Q}_a^\alpha, \mathfrak{Q}_b^\beta\} = \epsilon_{ab} \epsilon^{\alpha\beta} \frac{k}{2}, \quad \{\mathfrak{S}_\alpha^a, \mathfrak{S}_\beta^b\} = \epsilon^{ab} \epsilon_{\alpha\beta} \frac{k^*}{2} \quad (3.6)$$

These imply that the BPS condition reads $\epsilon^2 = 1 + k k^*$. For the fundamental bulk excitation we also have a relation between k and the momentum

$$|k|^2 = 16g^2 \sin^2 \frac{p}{2} \quad (3.7)$$

The phase of k is a bit more subtle and we will discuss it later.

The fundamental of $SU(2|2)$ can be split in the following way $\square = {}^B \square \oplus {}^F \square$, under $SU(2) \times SU(2)$, where we specified that one doublet is bosonic while the other one is fermionic, i.e. ${}^B \square = (\phi^+, \phi^-)$ and ${}^F \square = (\psi^+, \psi^-)$. We have added a dot to the bosonic

¹ In the notation of [1] $\frac{k}{2} = \mathfrak{P}$ and $\frac{k^*}{2} = \mathfrak{K}$.

$SU(2)$ indices to remind us that they transform under a different $SU(2)$ than the fermions. It is useful to write down the transformation rules for the fundamental multiplet as

$$\begin{aligned}\mathfrak{Q}_a^\alpha|\phi^b\rangle &= a\delta_a^b|\psi^\alpha\rangle, & \mathfrak{Q}_a^\alpha|\psi^\beta\rangle &= b\epsilon^{\alpha\beta}\epsilon_{ab}|\phi^b\rangle \\ \mathfrak{S}_a^\alpha|\phi^b\rangle &= c\epsilon_{\alpha\beta}\epsilon^{ab}|\psi^\beta\rangle, & \mathfrak{S}_a^\alpha|\psi^\beta\rangle &= d\delta_\alpha^\beta|\phi^a\rangle\end{aligned}\tag{3.8}$$

where $ad - cb = 1$. We find that $\frac{k}{2} = ab$, $\frac{k^*}{2} = cd$ and the energy is $\epsilon = 2\mathfrak{C} = ad + bc$. We will pick the following parametrization for (a, b, c, d) :

$$\begin{aligned}a &= \sqrt{g}\eta \\ b &= \frac{\sqrt{g}}{\eta}f\left(1 - \frac{x^+}{x^-}\right) \\ c &= \frac{\sqrt{g}i\eta}{fx^+} \\ d &= \frac{\sqrt{g}}{i\eta}(x^+ - x^-)\end{aligned}\tag{3.9}$$

The momentum of the particle is given by $\frac{x^+}{x^-} = e^{ip}$. The $ad - bc = 1$ condition translates into the mass shell condition

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}\tag{3.10}$$

The unitarity of the representation demands that

$$\eta = \sqrt{ix^- - ix^+}\tag{3.11}$$

up to a phase, which we set to one. Unitarity also requires that f is a phase, which contributes to the phase of the central charge as $k = -2gf(e^{ip} - 1)$. We can think about the central charges in terms of the segment that the magnon describes in the Z plane, by stretching from z_1 to z_2 ,

$$z_2 - z_1 = f(e^{ip} - 1) = -\frac{k}{2g}\tag{3.12}$$

Then the phase f represents the orientation of that segment, see figure 4. This orientation depends on the sum of the momenta of the magnons that are to the left of the magnon under consideration². Thus f is given by the angle that the magnon is making in a given state, relative to the magnon with the same momentum that starts at $z_1 = 1$ and goes to $z_2 = e^{ip}$, see figure 4. In the case that we have a semi-infinite string it is convenient to

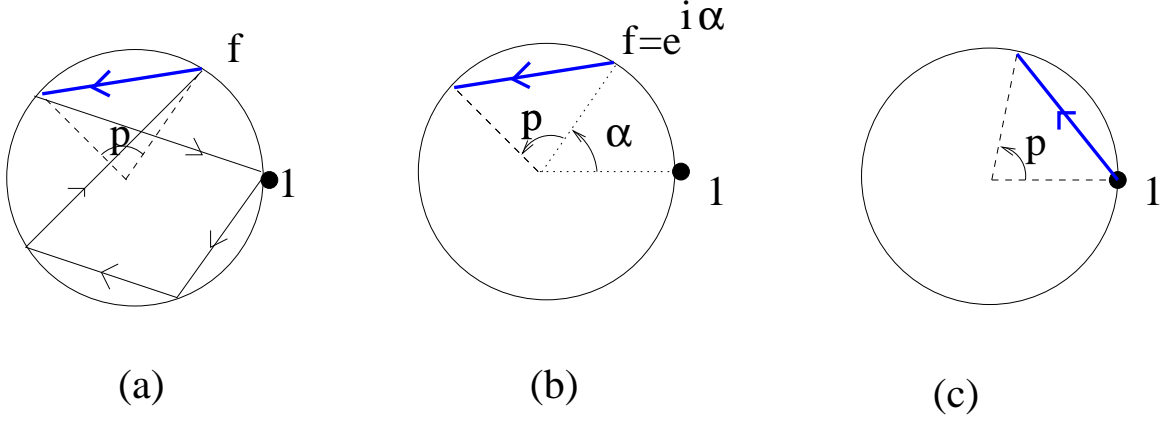


Figure 4: (a) We depict a configuration of well separated magnons living on a long string. We choose the point 1 as a reference point and we want to describe the magnon with momentum p . (b) f is the point on the unit circle where the magnon starts and gives the angle required to rotate it to the reference point 1, as in (c).

take the reference point to coincide with the point where this infinite string is located on the circle.

When we return to the full problem we need to consider two extended $SU(2|2)$ factors and the representation is the product of the fundamental for each, giving a total of 16 states. For example we get

$$Y = \phi^- \times \tilde{\phi}^-, \quad W = \phi^+ \times \tilde{\phi}^-, \quad \overline{W} = \phi^- \times \tilde{\phi}^+, \quad \overline{Y} = \phi^+ \times \tilde{\phi}^+ \quad (3.13)$$

where the fields ϕ^\pm and $\tilde{\phi}^\pm$ transform under two different $SU(2|2)$ groups. When we consider two extended $SU(2|2)$ factors we get six central charges. However, in this physical problem we require that the central charges for the two factors are equal (we set to zero the difference).

When we consider the $Z = 0$ giant graviton brane we preserve the full symmetry group. Physical states with finite J correspond to strings that start and end on the D-brane that sits at $Z = 0$ and they thus carry zero total central charges $k = k^* = 0$.

On the other hand when we consider the $Y = 0$ giant graviton brane we only preserve the subgroup which is also preserved by the field Y . Let us consider the anticommutator

$$\{\mathfrak{Q}_a^\alpha, \mathfrak{S}_\beta^b\} = \delta_a^b \delta_\beta^\alpha \mathfrak{C} + \delta_\beta^\alpha \mathfrak{K}_a^b + \delta_a^b \mathfrak{L}_\beta^\alpha \quad (3.14)$$

²This corresponds to the non-local parametrization of the problem, as described in [12]. This can also be described by forgetting about f and adding markers \mathcal{Z}^\pm , see [12] for details.

and concentrate on the supercharges with a $\dot{+}$ index, $\mathfrak{Q}^\alpha \equiv \mathfrak{Q}_{\dot{+}}^\alpha$ and $\mathfrak{S}_\alpha \equiv \mathfrak{S}_{\dot{+}\alpha}$. These supercharges annihilate an object with $\mathfrak{J} \equiv \mathfrak{C} + \mathfrak{R}_{\dot{+}}^{\dot{+}} = 0$, which is a singlet under the second $SU(2)$, such as a gauge invariant operator made purely with the field Y (notice that an upper $\dot{-}$ index carries $\mathfrak{R}_{\dot{+}}^{\dot{-}} = -\frac{1}{2}$). These supercharges, together with \mathfrak{J} and the second $SU(2)$ generators form an $SU(1|2)$ subgroup. The (noncompact) $U(1)$ generator³, \mathfrak{J} , in $SU(1|2)$, which appears in the right hand side of the supersymmetry algebra, is given by $2\mathfrak{J} = \epsilon + 2\mathfrak{R}_{\dot{+}}^{\dot{+}} = \Delta - J_{56} - J_{34} - J_{12}$ for one $SU(1|2)$ factor and it is $2\tilde{\mathfrak{J}} = \epsilon + 2\tilde{\mathfrak{R}}_{\dot{+}}^{\dot{+}} = \Delta - J_{56} - J_{34} + J_{12}$ for the other.

Let us now study each case in detail.

3.1 The $Y = 0$ giant graviton brane or $SU(1|2)^2$ theory

As we mentioned above, the symmetries that commute with the field Y lead to an $SU(1|2)^2$ subgroup. In order to study the problem we first focus on one $SU(1|2)$ subgroup and compute the reflection matrix in this case.

The $SU(1|2)$ algebra arises by restricting all the generators of the $SU(2|2)$ algebra to the ones carrying only $\dot{+}$ indices. As we mentioned above the (non-compact) $U(1)$ generator is $\mathfrak{J} = \mathfrak{C} + \mathfrak{R}_{\dot{+}}^{\dot{+}}$ and the non-vanishing commutators are

$$[\mathfrak{J}, \mathfrak{Q}^\alpha] = -\frac{1}{2}\mathfrak{Q}^\alpha \quad (3.15)$$

$$[\mathfrak{J}, \mathfrak{S}_\alpha] = \frac{1}{2}\mathfrak{S}_\alpha \quad (3.16)$$

$$[\mathcal{L}_\beta^\alpha, \mathcal{J}^\gamma] = \delta_\beta^\gamma \mathcal{J}^\alpha - \frac{1}{2}\delta_\beta^\alpha \mathcal{J}^\gamma \quad (3.17)$$

$$\{\mathfrak{Q}^\alpha, \mathfrak{S}_\beta\} = \mathcal{L}_\beta^\alpha + \delta_\beta^\alpha \mathfrak{J} \quad (3.18)$$

where \mathcal{J}^α is any generator with upper index α . Notice that this algebra is not centrally extended. All central extensions that appeared in the $SU(2|2)$ algebra do not contribute to the anticommutators of the surviving supercharges have disappeared. In this case a finite J physical open string does not necessarily have zero central charges, but the central charges, k , k^* are not preserved by the boundary.

We can find the action of this algebra on the states of the fundamental representation

³ This factor is really non-compact in our problem, hopefully we can continue to call it a $U(1)$ without causing confusion.

of $SU(2|2)$ from (3.8). For completeness we give the action of all generators

$$\begin{aligned}
\mathcal{L}_\beta^\alpha |\phi^\pm\rangle &= 0, & \mathcal{L}_\beta^\alpha |\psi^\gamma\rangle &= \delta_\beta^\gamma |\psi^\alpha\rangle - \frac{1}{2} \delta_\beta^\alpha |\psi^\gamma\rangle \\
\mathfrak{J} |\phi^\dot{-}\rangle &= bc |\phi^\dot{-}\rangle, & \mathfrak{J} |\phi^\dot{+}\rangle &= ad |\phi^\dot{+}\rangle, & \mathfrak{J} |\psi^\alpha\rangle &= \frac{1}{2} (ad + bc) |\psi^\alpha\rangle \\
\mathfrak{Q}^\alpha |\phi^\dot{-}\rangle &= 0, & \mathfrak{Q}^\alpha |\phi^\dot{+}\rangle &= a |\psi^\alpha\rangle, & \mathfrak{Q}^\alpha |\psi^\beta\rangle &= b \epsilon^{\alpha\beta} |\phi^\dot{-}\rangle \\
\mathfrak{S}_\alpha |\phi^\dot{-}\rangle &= c \epsilon_{\alpha\beta} |\psi^\beta\rangle, & \mathfrak{S}_\alpha |\phi^\dot{+}\rangle &= 0, & \mathfrak{S}_\alpha |\psi^\beta\rangle &= d \delta_\alpha^\beta |\phi^\dot{+}\rangle
\end{aligned}$$

with $\alpha, \beta, \gamma = +, -$.

Since the $SU(1|2)$ algebra does not have a central extension, we find that for general momentum we have a non-BPS representation since the charge $\mathfrak{J} = \frac{\epsilon}{2} + \mathfrak{R}_+^+$ can vary continuously. Thus we expect that the fundamental representation of the extended $SU(2|2)$ transforms irreducibly. In fact, it transforms as the representation of $SU(1|2)$ with the supertableaux $\square\square$. This has the right dimensions as $\square\square = {}^B 1_{\frac{\epsilon}{2}-\frac{1}{2}} \oplus {}^B 1_{\frac{\epsilon}{2}+\frac{1}{2}} \oplus {}^F \square_{\frac{\epsilon}{2}}$, where we have broken the representation in $U(1) \times SU(2)$ multiplets and we have indicated whether we have bosons or fermions. In terms of the degrees of freedom of the $SU(1|2)$ fundamental representation $\square = (\varphi, \chi^\pm)$ we can represent the corresponding states as $(\chi^+ \chi^- - \chi^- \chi^+, \varphi\varphi, \varphi\chi^\pm + \chi^\pm \varphi)$. We now would like to match these states to the fundamental of the extended $SU(2|2)$ algebra. Matching their bosonic charges we see that

$$\square\square = \begin{pmatrix} \chi^+ \chi^- - \chi^- \chi^+ \\ \varphi\varphi \\ \varphi\chi^- + \chi^- \varphi \\ \varphi\chi^+ + \chi^+ \varphi \end{pmatrix} = \begin{pmatrix} \phi^\dot{-} \\ \phi^\dot{+} \\ \psi^- \\ \psi^+ \end{pmatrix} \quad (3.19)$$

In the special case of zero momentum $p = 0$, the representation splits into two, one is the identity, given just by $\phi^\dot{-}$, and the other three states form the fundamental, BPS representation of $SU(1|2)$ with one bosonic, $\phi^\dot{+}$, and two fermionic states. Recall that the field Y is given by $Y = \phi^\dot{-} \times \tilde{\phi}^\dot{-}$, so it is reasonable that for zero momentum it is a singlet under $SU(1|2)$ since the $SU(1|2)$ subalgebra was found by demanding that all generators annihilate Y . In this article we are interested in the case with non-zero momentum where we have a single $SU(1|2)$ non-BPS representation.

3.1.1 The reflection matrix

The $SU(1|2)$ reflection matrix⁴ \mathcal{R} can now be calculated by demanding that $[\mathcal{R}, \mathcal{J}] = 0$ for all generators \mathcal{J} . The vanishing of the commutators of \mathcal{R} and the bosonic operators imply that \mathcal{R} must be diagonal with equal entries for the fermionic components. Namely,

$$\mathcal{R} = \begin{pmatrix} r^- & 0 & 0 & 0 \\ 0 & r^+ & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \end{pmatrix} \quad (3.20)$$

The commutators with the fermionic operators yield the following conditions:

$$\begin{aligned} ar - a'r^+ &= 0 \\ br^- - b'r &= 0 \\ cr - c'r^- &= 0 \\ dr^+ - d'r &= 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} r^- &= \frac{c}{c'}r = \frac{b'}{b}r \\ r^+ &= \frac{a}{a'}r = \frac{d'}{d}r \end{aligned} \quad (3.21)$$

where the primed variables are the quantum numbers of the state after the reflection. These are obtained from the original ones by

$$x^\pm \rightarrow x'^\pm = -x^\mp \quad (3.22)$$

This follows from conservation of energy, $p \rightarrow -p$ and holding $x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}$. Note that η , (3.11), is invariant under (3.22), so $\eta' = \eta$. The phase f might change as well. f represents the point where the magnon starts in the Z circle, see figure 4. When we have a boundary scattering process the values for f for the incoming and the outgoing magnon are related by the geometry of the scattering process in the Z plane. In other words, it is determined by the conservation laws. We represent the relevant conservation laws in figure 5 for the scattering from a right boundary and a left boundary.

We see that in the case that we scatter from a boundary on the right, then f does not change, $f' = f$. If the orientation is opposite (boundary on the left), f changes to $f' = f \left(\frac{x^+}{x^-} \right)^2$, see figure 5(c,d). Incidentally, (3.21) requires $bc = b'c'$ and $ad = a'd'$. This follows trivially from conservation of energy $\epsilon = ad + bc$ and the mass shell condition

⁴The full reflection matrix of the theory is just the product of two $SU(1|2)$ reflection matrices.

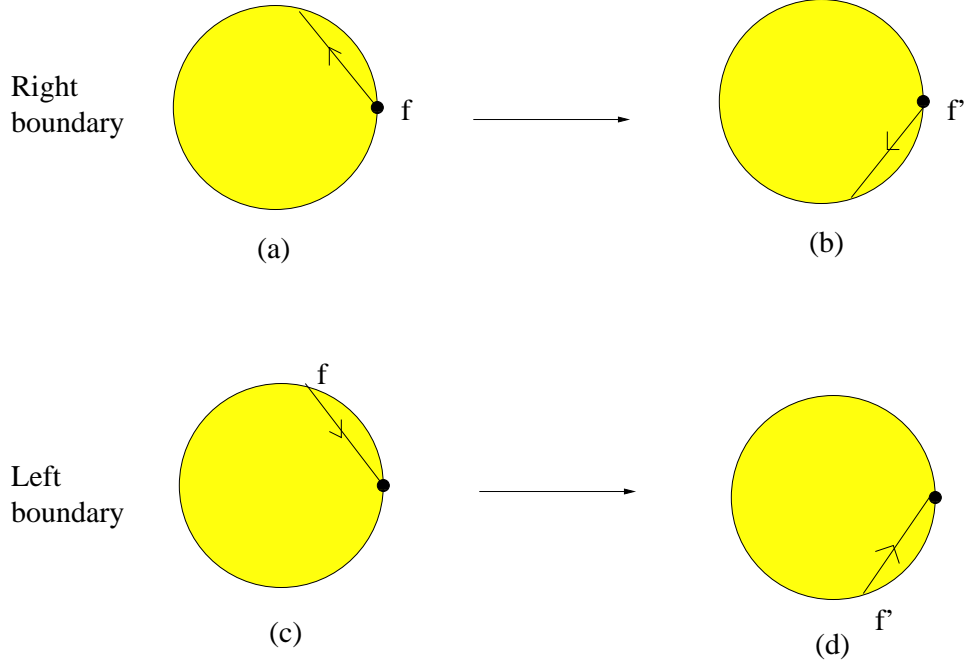


Figure 5: We depict several scattering configurations in a situation where we have a semi-infinite string. We choose the infinite region (“heavy” particle / BMN vacuum) to lie at the reference point 1 in the complex plane. We can read off the values of the phase f for the initial and final states from these figures. In (a) and (b) we depict the initial and final configuration for the scattering off a boundary on the right. We can see that in this case $f = f' = 1$. In (c) and (d) we have the initial and final configurations for scattering from a boundary on the left. $f = e^{-ip} \neq f' = e^{+ip}$ in this setup. In all cases we located the point that sets the phase for the incoming state, f , and for the final state, f' . The arrow goes from left to right on the string worldsheet.

$ad - bc = 1$. Plugging in the values for the quantum numbers yields

$$\mathcal{R}_R = \mathcal{R}_{0R}(p) \begin{pmatrix} -e^{-ip} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{for a right boundary} \quad (3.23)$$

and

$$\mathcal{R}_L = \mathcal{R}_{0L}(p) \begin{pmatrix} -e^{ip} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{for a left boundary.} \quad (3.24)$$

In these expressions $\frac{x^+}{x^-} = e^{ip}$ and $\mathcal{R}_{0R}(p)$, $\mathcal{R}_{0L}(p)$ are arbitrary phases. We see that the two results are consistent with the reflection symmetry that we have in the problem. In fact, if we assume reflection symmetry we can also relate $\mathcal{R}_{0L}(p) = \mathcal{R}_{0R}(-p)$. In addition, unitarity requires $\mathcal{R}_{0L}(-p) = 1/\mathcal{R}_{0L}(p)$, $\mathcal{R}_{0R}(-p) = 1/\mathcal{R}_{0R}(p)$.

The magnons in the full theory are the product of two fundamental magnons of each extended $SU(2|2)$ algebra. Similarly, they are the product of representations for each $SU(1|2)$ subalgebra.

From this result we can predict a ratio of reflection amplitudes. For example the ratio of the amplitudes of scattering a $Y = \phi^{\dot{-}} \times \tilde{\phi}^{\dot{-}}$ and a $W = \phi^{\dot{+}} \times \tilde{\phi}^{\dot{-}}$ is $-e^{\mp ip}$ for R, L boundaries respectively. Remember that in our conventions p is the incoming momentum. If the boundary is placed on the left this momentum is negative. So left and right results are consistent. We will compare this result with explicit calculations in the following sections.

Another interesting comment is that this matrix does not contain poles or zeros, unless they are included explicitly in $\mathcal{R}_0(p)$. This means that if there is a bound state in one channel, all channels must have one. In the next section we will check that there is no bound state at weak coupling. We will also compute $\mathcal{R}_0(p)$ perturbatively to two loops at weak coupling and to leading order at strong coupling.

3.1.2 The Yang Baxter equation

We now check that this reflection matrix satisfies the boundary Yang Baxter equation. This equation is represented graphically in figure 6 and it states that one can compute

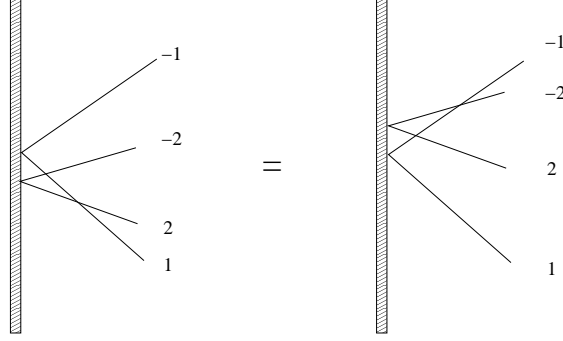


Figure 6: The content of the Yang Baxter equation is that these two processes give the same answer.

the reflection of a pair of particles in two ways. As in the case of the bulk Yang Baxter equation one can check the equation in a simple way using the symmetries [12]. The idea is to look at the Hilbert space of two particles and decompose it in representations of $SU(1|2)$ and then check the equation in each representation. This can be done in a simple way if each representation contains a state that scatters diagonally, so that all scattering amplitudes are simply phases. The intermediate representations of the 2 particle incoming states are:

$$\begin{array}{|c|c|} \hline \diagup & \diagdown \\ \hline \end{array} \times \begin{array}{|c|c|} \hline \diagup & \diagdown \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline \diagup & \diagdown & \diagup & \diagdown \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \diagup & \diagdown & \diagup \\ \hline \end{array} + \begin{array}{|c|c|} \hline \diagup & \diagdown \\ \hline \end{array} \quad (3.25)$$

The first representation on the right hand side of (3.25) contains the state $\phi_1^+ \phi_2^+$, the second contains the states $\psi_1^+ \psi_2^+$ and $\psi_1^- \psi_2^-$ and the third one contains $\phi_1^- \phi_2^-$, which are all states that scatter diagonally.

Let us now check the boundary Yang Baxter equation for two excitations that scatter diagonally. Let us denote by $S(1, 2)$ their bulk scattering. $S(1, 2)$ is simply a phase by assumption. Similarly, we have the reflection $r(1)$ and $r(2)$ from the boundary which is also a phase. Thus we have

$$S(1, 2)r(1)S(-2, 1)r(2) = r(2)S(-1, 2)r(1)S(-2, -1) \quad (3.26)$$

Since we only have phases we see that $r(1)$ and $r(2)$ drop out from the equation and we are only left with a requirement involving the bulk S matrix. This requirement is obeyed if the bulk S matrix is parity invariant, $S(1, 2) = S(-2, -1)$. This is an invariance of the bulk S matrix, thus we see that the boundary Yang Baxter equation is satisfied. We have also checked explicitly that the equation is indeed satisfied.

3.1.3 The crossing equation

In order to derive the crossing equation we need to form a singlet state according to the derivation in [12]. This identity state is

$$\mathbf{1}_{(p,\bar{p})} = f_p e^{ip/2} (\phi_p^+ \phi_{\bar{p}}^- - \phi_p^- \phi_{\bar{p}}^+) + \epsilon_{\alpha\beta} \psi_p^\alpha \psi_{\bar{p}}^\beta \quad (3.27)$$

where the subindex p denotes the momentum and energy $\epsilon(p)$ of the first particle and the index \bar{p} denotes the momentum $\bar{p} = -p$ and energy $\bar{\epsilon} = -\epsilon(p)$ of the second, crossed, particle. If we think in terms of the fermionic part of the state we can view the state as a hole, $\psi_+(p)$, and negative energy electron $\psi_-(\bar{p})$. In this case, we clearly see that we get back the original vacuum of the theory. Thus adding this state should have no effect on the theory. By scattering this two particle state from a third and demanding that the result is invariant one can obtain the crossing equation [2, 12].

If we start with this state and we scatter it from the right boundary we obtain the state $r(p)\mathbf{1}_{(-\bar{p},-p)}$, where $r(p)$ is some reflection phase. We see that we do not get the same state because the particle and antiparticle are in a different order. However, if we have a left boundary and we now scatter the resulting state we get back to the original state (3.27), see figure 7. We now use that parity invariance implies that the scattering phase we get from the second scattering is the same as the one we got from the first boundary. Thus we find that the total scattering phase is $r(p)^2$. Now it makes sense to demand that the total scattering phase is one, $r(p)^2 = 1$.

So, we get $r(p) = \pm 1$. By considering different boundaries on the two sides we see that the signs should be all plus or all minus, for all boundaries in the theory. We take this sign to be plus. We'll show this in a moment, by looking at the plane wave limit.

When we scatter this state from the boundary we will need the boundary reflection matrix (3.23) and the bulk S matrix written in [12].

At the end of the day we obtain

$$\begin{aligned} \mathbf{1}_{(p,\bar{p})} &= h_b S_0(p, -\bar{p}) \mathcal{R}_{0R}(p) \mathcal{R}_{0R}(\bar{p}) \left[f_{-p} e^{ip/2} (\phi_{-\bar{p}}^+ \phi_{-p}^- - \phi_{-\bar{p}}^- \phi_{-p}^+) + \epsilon_{\alpha\beta} \psi_{-\bar{p}}^\alpha \psi_{-p}^\beta \right] = \\ &= h_b S_0(p, -\bar{p}) \mathcal{R}_{0R}(p) \mathcal{R}_{0R}(\bar{p}) \mathbf{1}_{(-\bar{p},-p)} \\ h_b &\equiv \frac{\frac{1}{x^-} + x^-}{\frac{1}{x^+} + x^+} \end{aligned} \quad (3.28)$$

where S_0 is the phase factor as defined by Beisert in [12] and \mathcal{R}_0 is the phase factor which multiplies the boundary reflection matrix that we had above. Thus the crossing equation

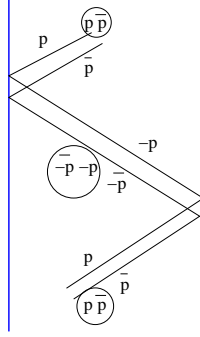


Figure 7: We scattering the singlet state $p\bar{p}$ from the right boundary and then from the left boundary in order to come back to the original situation. We demand that this double scattering gives one.

has the form

$$\mathcal{R}_{0R}(p)\mathcal{R}_{0R}(\bar{p}) = \frac{1}{h_b} \frac{1}{S_0(p, -\bar{p})} = \frac{\frac{1}{x^+} + x^+}{\frac{1}{x^-} + x^-} \frac{1}{S_0(p, -\bar{p})} \quad (3.29)$$

This would be the equation in the case that we had only one $SU(2|2)$ factor. In the full theory, where we have the two $SU(2|2)$ factors we define the full reflection factor to be simply $\mathcal{R}_{0R}^2(p)$, and the bulk phase factor is usually written in terms of a dressing factor σ^2 through the equation [7]

$$S_0(p_1, p_2)^2 = \frac{(x_1^+ - x_2^-)}{(x_1^- - x_2^+)} \frac{(1 - \frac{1}{x_1^- x_2^+})}{(1 - \frac{1}{x_1^+ x_2^-})} \frac{1}{\sigma^2(p_1, p_2)} \quad (3.30)$$

Then the equation for the full theory becomes

$$\mathcal{R}_{0R}^2(p)\mathcal{R}_{0R}^2(\bar{p}) = \frac{1}{h_b^2} \frac{1}{S_0^2(p, -\bar{p})} = \frac{x^+ + \frac{1}{x^+}}{x^- + \frac{1}{x^-}} \sigma^2(p, -\bar{p}) \quad (3.31)$$

Notice that in the plane wave limit [9] the right hand side of this equation is just 1. In this limit our theory is non interacting and we know that, in the $SU(2)$ subsector, $\mathcal{R}_R^2(p) = \mathcal{R}_R^2(\bar{p}) = -1$, as this is just a relativistic theory with Dirichlet boundary conditions. From equation (3.23) we see that this implies $\mathcal{R}_{0R}^2(p) = \mathcal{R}_{0R}^2(\bar{p}) = -1$. This means that the plus sign is the correct one for the right hand side of equation (3.31).

Finally, we should also mention that unitarity implies

$$\mathcal{R}_{0R}(p)\mathcal{R}_{0R}(-p) = 1 \quad (3.32)$$

3.2 The $Z = 0$ giant graviton brane or $SU(2|2)^2$ theory

We now study the case of a $Z = 0$ giant graviton brane, which preserves the full $SU(2|2)$ symmetry, see figures 1, 3. The new feature of this case is the existence of a boundary degree of freedom. We assume that the boundary degree of freedom transforms in the fundamental representation of extended $SU(2|2)^2$. It seems clear that this is the case at weak coupling where we have an impurity stuck between the Z -determinant and the string of Z 's producing the large J open string. Then we expect that this should continue to be the case at all values of the coupling. Since the supersymmetry algebra has been extended by the addition of two central charges we need to understand the values of the central charges for the impurity. Here, we will be guided by the string pictures we discussed above, where the central charges are associated to the winding number of the string in the z plane. Thus the central charge vector is simply the vector given by a string going from the brane at $z = 0$ to the rim of the disk, see figure 3 (c). We can also view the central charge vector as a complex number. This fixes the absolute value of the central charge vector

$$|k|^2 = 4g^2 \tag{3.33}$$

The phase of the central charge depends on the momenta of the other magnons that are in the problem and changes when a magnon scatters from the boundary. Below we will explain how it changes. The conclusion is that the representation of the boundary impurity is again the fundamental of the extended symmetry algebra. The only difference between the impurity representation and the magnon one is in the relation between the central charges and the momentum (the impurity does not have a momentum quantum number), and in the precise dynamics of the phase of the central charge. It turns out that the problem completely factorizes into each extended $SU(2|2)$ factor. Thus we consider first the case where we have only one $SU(2|2)$ factor.

Let us start by being more specific about the representation properties of the boundary degree of freedom. The transformation properties are as in the bulk case, (3.8), but with

the following values of a, b, c, d .

$$a_B = \sqrt{g}\eta_B \quad (3.34)$$

$$b_B = \frac{\sqrt{g}f_B}{\eta_B} \quad (3.35)$$

$$c_B = \frac{\sqrt{g}i\eta_B}{x_B f_B} \quad (3.36)$$

$$d_B = \frac{\sqrt{g}x_B}{i\eta_B} \quad (3.37)$$

where we have added the subindex B to distinguish these from the bulk case. Unitarity of the representation requires $|\eta_B|^2 = -ix_B$ and that f_B is just a phase. The shortening/mass shell condition implies

$$ad - bc = 1 \quad \longrightarrow \quad x_B + \frac{1}{x_B} = \frac{i}{g}, \quad x_B = \frac{i}{2g} \left(1 + \sqrt{1 + 4g^2}\right) \quad (3.38)$$

where we picked the solution for x_B which leads to positive energy

$$\epsilon = ad + bc = \frac{g}{i} \left(x_B - \frac{1}{x_B}\right) = \sqrt{1 + 4g^2} \quad (3.39)$$

The phase f_B depends on the other magnons in the problem and can be understood most simply by looking at figure 8. For a right boundary, f_B is the position of the endpoint of the last magnon on the Z circle. Equivalently it is given by the sum of the momenta of all magnons to the left of the boundary. Since the system ends at the right boundary, this means that $f_B = \prod_j e^{ip_j} f_1$ for all the magnons in the system, where f_1 is the starting point of the first magnon.

We now derive the boundary S matrix for this system. We must first understand how f and f_B change under scattering, see figure 9. Let us consider the case of right boundary scattering. In the initial state we have $f_B = e^{ip}f$. In the final state the magnon phase does not change, $f' = f$ and $f'_B = e^{-ip}f = e^{-2ip}f_B = \left(\frac{x^-}{x^+}\right)^2 f_B$, see figure 9 a, b. On the other hand, for a left boundary $f_B = -f$, see figure 9 c, d. In this case $f' = -f'_B = e^{2ip}f$, or $f'_B = \left(\frac{x^+}{x^-}\right)^2 f_B$. x_B does not change in either case.

Let us now analyze the case with a left boundary in detail. The following equations summarize the quantum numbers of the incoming particle and the boundary and how

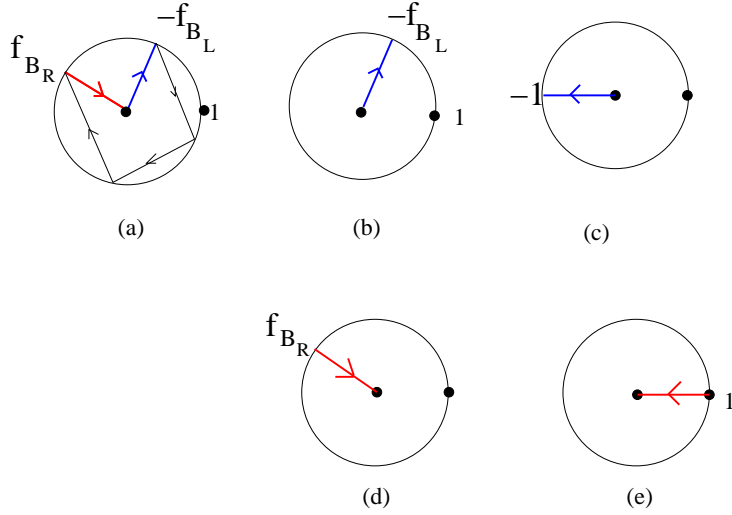


Figure 8: In (a) we see a generic open string configuration in the regime that J is very large and the magnons are very well separated. We have denoted by f_{B_L} and f_{B_R} the corresponding parameters of left and right boundaries, respectively. In (b) we isolate the piece of string corresponding to the left boundary impurity. Its phase $-f_B$ is the end point of this string. f_B is also the phase by which the configuration was rotated with respect to the reference configuration in (c). In (d) we isolated the piece of string corresponding to the right boundary. f_B is the starting point of the string on the circle. This phase is also the one by which the configuration was rotated with respect to the reference configuration in (e). These figures can be viewed as the central charge vectors (except for a $-2g$ factor) for the states involved and also as the projections of the physical string configurations to the z plane in the $AdS_5 \times S^5$ geometry.

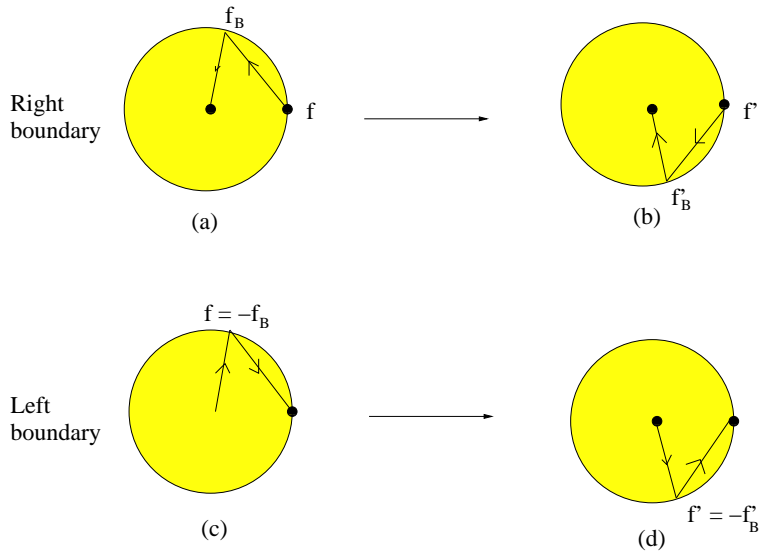


Figure 9: In (a) we see the initial state for scattering from a right boundary and in (b) we see the final state. We have indicated the phases of the central charge in both cases. In (c) we see the initial state for right boundary scattering and in (d) we see the final state. These figures can be viewed as the central charge vectors (except for a $-2g$ factor) for the states involved and also as the projections of the physical string configurations to the z plane in the $AdS_5 \times S^5$ geometry.

they change after scattering:

$$\begin{aligned}
a &= \sqrt{g}\eta & a &= a \\
b &= \frac{\sqrt{g}}{\eta}f\left(1 - \frac{x^+}{x^-}\right) & b' &= -\frac{x^+}{x^-}b \\
c &= \frac{\sqrt{g}i\eta}{fx^+} & c' &= -\frac{x^-}{x^+}c \\
d &= \frac{\sqrt{g}}{i\eta}(x^+ - x^-) & d' &= d
\end{aligned} \tag{3.40}$$

$$\begin{aligned}
a_B &= \sqrt{g}\eta_B & a'_B &= a_B \\
b_B &= -\frac{\sqrt{g}f}{\eta_B} & b'_B &= \left(\frac{x^+}{x^-}\right)^2 b_B \\
c_B &= -\frac{\sqrt{g}i\eta_B}{x_B f} & c'_B &= \left(\frac{x^-}{x^+}\right)^2 c_B \\
d_B &= \frac{\sqrt{g}x_B}{i\eta_B} & d'_B &= d_B
\end{aligned} \tag{3.41}$$

In order to calculate the reflection matrix, \mathcal{R} , we demand that all commutators of the reflection matrix with the generators of $SU(2|2)$ vanish. In this case the operators act on two particle states, so the computation is more involved than in the last case. In particular, we have to remember that fermionic operators acting on two particle states are defined as $Q = Q_1 \otimes 1 + (-)^F \otimes Q_2$, where F is the fermionic number of particle state 1. The computation is almost identical to the one performed in [1]. Invariance under the bosonic generators implies that the \mathcal{R} matrix can be written as [1][12]

$$\mathcal{R}|\phi_B^a \phi_p^b\rangle = A|\phi_B^{\{a} \phi_{-p}^{b\}}\rangle + B|\phi_B^{[a} \phi_{-p}^{b]}\rangle + \frac{1}{2}C\epsilon^{ab}\epsilon_{\alpha\beta}|\psi_b^\alpha \psi_{-p}^\beta\rangle \tag{3.42}$$

$$\mathcal{R}|\psi_B^\alpha \psi_p^\beta\rangle = D|\psi_B^{\{\alpha} \psi_{-p}^{\beta\}}\rangle + E|\psi_B^{[\alpha} \psi_{-p}^{\beta]}\rangle + \frac{1}{2}F\epsilon^{\alpha\beta}\epsilon_{ab}|\phi_B^a \phi_{-p}^b\rangle \tag{3.43}$$

$$\mathcal{R}|\phi_B^a \psi_p^\alpha\rangle = G|\psi_B^\alpha \phi_{-p}^a\rangle + H|\phi_B^a \psi_{-p}^\alpha\rangle \tag{3.44}$$

$$\mathcal{R}|\psi_B^\alpha \phi_p^a\rangle = K|\psi_B^\alpha \phi_{-p}^a\rangle + L|\phi_B^a \psi_{-p}^\alpha\rangle \tag{3.45}$$

where a, b represent bosonic indices, \pm , and α, β are fermionic indices, \pm . The (anti)symmetrization symbols are defined with a $\frac{1}{2}$ normalization factor, i.e. $\{ab\} = \frac{ab+ba}{2}$.

It is understood that the states on the right hand side of these equations are out states and, therefore, have primed quantum numbers. In particular, they have primed phases, f' and f'_B .⁵

Acting with the fermionic generators on both sides we get constraints on A , B , C ,

⁵Note that we are working in the so called non-local representation [12]. One can also reintroduce the markers \mathcal{Z}^\pm in a simple way.

D, E, F, G, H, K, L that determine them completely up to an overall phase. We get:

$$\begin{aligned}
A &= \mathcal{R}_0 \frac{x^+(x^+ + x_B)}{x^-(x^- - x_B)} \\
B &= \mathcal{R}_0 \frac{2x^+x^-x_B + (x^+ - x_B)[-2(x^+)^2 + 2(x^-)^2 + x^+x^-]}{(x^-)^2(x^- - x_B)} \\
C &= \mathcal{R}_0 \frac{2\eta\eta_B}{f} \frac{(x^- + x^+)(x^-x_B - x^+x_B - x^-x^+)}{x_Bx^-(x^+)^2(x^- - x_B)} \\
D &= \mathcal{R}_0 \\
E &= \mathcal{R}_0 \frac{2[(x^+)^2 - (x^-)^2][-x^+x^- + x_B(x^- - x^+ + x^-(x^+)^2)] - x_B(x^+x^-)^2(x_B - x^-)}{(x^-x^+)^2x_B(x^- - x_B)} \\
F &= \mathcal{R}_0 \frac{2f}{\eta\eta_B} \frac{(x^+ - x^-)(x^+ + x^-)(x_Bx^+ - x_Bx^- + x^+x^-)}{(x^-)^3(x^- - x_B)} \\
G &= \mathcal{R}_0 \frac{\eta_B}{\eta} \frac{(x^+ - x^-)(x^+ + x^-)}{(x^- - x_B)x^-} \\
H &= \mathcal{R}_0 \frac{(x^+)^2 - x_Bx^-}{x^-(x^- - x_B)} \\
K &= \mathcal{R}_0 \frac{[x_Bx^+ + (x^-)^2]}{x^-(x^- - x_B)} \\
L &= \mathcal{R}_0 \frac{\eta}{\eta_B} \frac{(x^+ + x^-)x_B}{x^-(x^- - x_B)}
\end{aligned} \tag{3.46}$$

Notice that the phase f appears explicitly in C and F . We can eliminate f at the cost of introducing markers, \mathcal{Z}^\pm , as explained in [12].

The boundary Yang Baxter equation is satisfied by the exactly the same argument used by Beisert in [12], as the symmetries and representations are the same as in the bulk. As in that case, there are two intermediate representations for 3 particle states and each one contains a state that scatters diagonally.

Note also that the boundary scattering in the full theory is given by taking the product of two such reflection matrices, one for each $SU(2|2)$ factor. One could also derive a crossing equation by scattering the identity state (3.27) as we did in the $SU(1|2)$ case.

Note that $\frac{A}{B}$ is a prediction for the ratio of amplitudes of $YY \rightarrow YY$ scattering in the $SU(2)$ sector to $\psi\psi \rightarrow \psi\psi$ in the $SU(1|1)$ sector. In the following section we will test the ratio $\frac{A}{B}$ and calculate the phase factor at weak coupling.

3.2.1 Boundary bound states

It is interesting to note that the coefficient A has a pole at $x^- = x_B$. In the full problem, once we take the product of the two reflection matrices we expect that the overall phase



Figure 10: Pole at $x^- = x_B$

factor is such that the scattering in the $SU(2)$ subsector continues to have a single pole at this position. In fact, this will be explicitly checked at weak coupling in section 4.3. Thus, we expect to have single pole at all values of the coupling. This pole signals the presence of a bound state, similar to the ones considered in [37]. Following the same rules as in [46] we see that this pole is generated by the Landau diagram in figure 10 that yields a normalizable wave function. Figure 10 represents an actual boundary bound state in the s-channel. The incoming fundamental magnon binds to the boundary degree of freedom to form a BPS bound state corresponding to a double box representation of $SU(2|2)^2$. As in the bulk case, we can introduce a new parameter $x_B^{(2)} \equiv x^+$. Once we set $x^- = x_B$, we find that

$$x_B^{(2)} + \frac{1}{x_B^{(2)}} = 2\frac{i}{g} \quad (3.47)$$

The energy of the bound state is given by $\epsilon = \frac{g}{i}(x_B^{(2)} - \frac{1}{x_B^{(2)}})$, as in (3.39). We can now consider the boundary scattering of another magnon with this new boundary impurity. This can be computed by scattering this second magnon, parametrized by x_2^\pm , off the bound state made out of the original impurity and the first magnon, parametrized by $x_B^{(2)} = x_1^+$, $x^- = x_B$. This scattering is described as the product of the scattering amplitudes of the second magnon from the first, the reflection matrix, and the scattering of the reflected second magnon with the first. This full amplitude has a pole at $x_2^- = x_B^{(2)}$. Thus we can have a new bound state characterized by $x_B^{(3)} \equiv x_2^+$. Proceeding in this fashion we obtain a structure of bound states very similar to what we had in the bulk [37, 56]. An n particle bound state is given by $x_B = x_1^-$, $x_1^+ = x_2^-$, $x_i^+ = x_{i+1}^-$, $x_B^{(n)} = x_{n-1}^+$. Then using

the equations for each of the particles one can see that

$$x_B^{(n)} + \frac{1}{x_B^{(n)}} = n \frac{i}{g}, \quad \epsilon_B = \frac{g}{i} \left(x_B^{(n)} - \frac{1}{x_B^{(2)}} \right) = \sqrt{n^2 + 4g^2} \quad (3.48)$$

These are in the same representation of the extended $SU(2|2)^2$ superalgebra as the bulk magnons [38], except, of course, that the central charges are given by the line going from the center of the disk to the rim of the disk.

4 Results at weak coupling

In this section we present some results obtained from weak coupling calculations in the gauge theory. We consider the operators \mathcal{O}_Y and \mathcal{O}_Z described by expressions (2.3) and (2.4). We study the large J limit, where the chain is infinitely long and we focus on the physics near each of the boundaries. We study $\mathcal{N} = 4$ super Yang Mills at two loops, using the results for the dilatation operator obtained in [47] to calculate the reflection matrices in the $SU(2)$ subsector. Furthermore, we perform some non trivial checks, in the $SU(3)$ subsector, of the ratios of the matrix elements of the exact matrices discussed in the previous section. Finally, in appendix A we discuss the integrability of the resulting Hamiltonian.

4.1 The two loop Hamiltonian at weak coupling in the $SU(2)$ sector

In order to calculate the reflection matrices we first need to calculate the appropriate Hamiltonian including the boundary contributions. This has been calculated at one loop in [23] and at two loops in [31]. We review this calculation and discuss an extra term, relative to [31], that is present at two loops. This term, although subtle, is crucial to make the spin chain integrable.

Our starting point is the general expression for the one and two loop dilatation operator [47] in the $SU(2)$ subsector. This is

$$\begin{aligned} D = & -2 \frac{g^2}{N} : \text{Tr}[Y, Z][\check{Y}, \check{Z}] : - 2 \frac{g^4}{N^2} : \text{Tr}[[Y, Z], \check{Z}][[\check{Y}, \check{Z}], Z] : \\ & - 2 \frac{g^4}{N^2} : \text{Tr}[[Y, Z], \check{Y}][[\check{Y}, \check{Z}], Y] : + 4 \frac{g^4}{N} : \text{Tr}[Y, Z][\check{Y}, \check{Z}] : \end{aligned} \quad (4.49)$$

where \check{X} means $\frac{\partial}{\partial X}$.

We can calculate the effective Hamiltonian operating on a $SU(2)$ spin chain from this operator. The bulk part of this Hamiltonian is [31, 47]

$$H_{\text{bulk}} = \sum_i (2g^2 - 8g^4)(I - P_{i,i+1}) + 2g^4 \sum (I - P_{i,i+2}) \quad (4.50)$$

where $P_{i,j}$ is the permutation operator between sites i and j .

Let us discuss the boundary terms that need to be added when we attach our spin chain to a giant graviton. As the interaction has a range of two sites we only need to worry about the first few sites of the chain, assuming a boundary on the left. Let us assume our spin chain starts as

$$\epsilon X_B^{N-1} | \underbrace{X_0}_0 \underbrace{X_1}_1 \underbrace{X_2}_2 \dots \quad (4.51)$$

where X_i are fields that can take the values Y, Z . We have been schematic and have omitted indices in this expression. The $|$ separates the giant graviton from the rest of the chain.

From the site 1 onwards we have the bulk Hamiltonian. At site 0, the Hamiltonian acts differently. To leading order in $1/N$, the determinant cannot have a field of the same flavor next to it [23, 24, 43]. This means that X_0 is always different from X_B . We also have to be careful about this when we operate with the Hamiltonian. If X_1 or X_2 are equal to X_B then the corresponding permutation operator acting on the site 0 will vanish. With these rules in mind, if we consider the action of D (4.49) on the chain by applying all derivatives outside the determinant⁶, we find that H acts on the first three sites as

$$H_{\text{naive}} = (2g^2 - 8g^4)q_1^{X_B} + 2g^4 q_2^{X_B} \quad (4.52)$$

where $q_i^{X_B}$ acts as the identity if $X_i = X_B$ and as zero if it is not. If this was the whole story we would reproduce the results of [31]. However, we still need to consider the possibility of the dilatation operator acting on the determinant and its neighboring sites. It turns out there is only one term in the dilatation operator (4.49) that contributes to this extra piece. This term is roughly $\frac{g^4}{N^2} \text{Tr} \left(\check{X}_B X_0 X_B \check{X}_B \check{X}_0 X_B \right)$ with the first derivative acting on the determinant. Naively, this term is suppressed by a factor of N as can be seen from (4.49). However, since there are $N - 1$ letters inside the determinant, there are $\mathcal{O}(N)$ possible actions of the derivative. All these subleading terms add up cancelling the

⁶This amounts to truncating H_{bulk} at the end of the chain.

$\frac{1}{N}$ suppression. This extra term is⁷

$$H_{\text{det}} = 4g^4 q_1^B \quad (4.53)$$

The final form of the two loop boundary Hamiltonian in the $SU(2)$ sector is:

$$\begin{aligned} H &= H_{\text{bulk}} + H_{\text{naive}} + H_{\text{det}} = \\ &= (2g^2 - 8g^4) \sum_{i=1}^{\infty} (I - P_{i,i+1}) + 2g^4 \sum_{i=1}^{\infty} (I - P_{i,i+2}) + (2g^2 - 4g^4) q_1^{X_B} + 2g^4 q_2^{X_B} \end{aligned} \quad (4.54)$$

Notice that the chain starts effectively at site 1, as the site 0 is fixed by the boundary⁸. This Hamiltonian, with the explicit inclusion of H_{det} (4.53), is consistent with integrability. This is suggested in appendix A by explicitly constructing the perturbative asymptotic Bethe ansatz solution for the two magnon problem.

We can now use this result to calculate scattering amplitudes for different boundaries in the $SU(2)$ subsector.

4.2 The $SU(1|2)$ reflection matrix off a $\det(Y)$ boundary

Let us now consider the operators involving an open chain on ending on the operator $\det(Y)$, corresponding to the $Y = 0$ giant graviton brane. We focus on the large J limit, where we have a large number of Z s producing a long open string, and we focus on one end of the chain at a time. In that case one can compute the boundary reflection matrix. Let us start considering the operator \mathcal{O}_Y (2.2) that corresponds to the vacuum. Acting with the Hamiltonian (4.54) we find

$$H\mathcal{O}_Y(Z) = 0 \quad (4.55)$$

where we plugged in $X_B = Y$ in the expression (4.54). This was expected, since it is a BPS state the vacuum has zero energy. We see that we have no degree of freedom, as the first excitations will be massive. If we place an impurity moving with momentum p far away from the boundary, all boundary terms vanish, and we recover the bulk expression for the energy

$$H\mathcal{O}_Y(Y_p) = \left(8g^2 \sin^2 \frac{p}{2} - 32g^4 \sin^4 \frac{p}{2} \right) \mathcal{O}_Y(Y_p) \quad (4.56)$$

⁷This term should also be added to the expressions in [30].

⁸This situation will change when we move to the $SU(3)$ subsector

for a one particle state with momentum p , $\mathcal{O}_Y(Y_p)$; see equation (4.58). The formula for the energy is just the expansion to second order in g^2 of the anomalous part of the magnon energy

$$\epsilon - 1 = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}} - 1 \sim 8g^2 \sin^2 \frac{p}{2} - 32g^4 \sin^4 \frac{p}{2} \quad (4.57)$$

Let us now compute the reflection matrix. We write a wavefunction of the form

$$\mathcal{O}_Y(Y_p) = \sum_{x=1}^{\infty} \Psi(x) \mathcal{O}_Y(Y_x) = \sum_{x=1}^{\infty} (e^{+ipx} + Re^{-ipx}) \mathcal{O}_Y(Y_x) \quad (4.58)$$

where $\mathcal{O}_Y(Y_x)$ is an operator of the form given by equation 2.3 with the impurity placed at site x . In principle, there can be corrections of order g^2 near $x \sim 0$, as was discussed for the bulk in [10]. This turns out not to be necessary in our case. If we apply the Hamiltonian we see that this is an eigenstate of the right energy, provided we set

$$\Psi(0) = 0 \quad \Psi(-1) + \Psi(1) = 0 \quad (4.59)$$

where we have analytically continued the expression for the wavefunction, $\Psi(x) = e^{ipx} + Re^{-ipx}$, to negative values of x . Remarkably both equations can be satisfied simultaneously without the inclusion of corrections by setting $R = -1$. In terms of the reflection matrix for each $SU(1|2)$ factor (3.24), and recalling the expression for Y , (3.13), we see that

$$-1 = R = \mathcal{R}_{0L}^2 e^{2ip}, \quad \rightarrow \quad \mathcal{R}_{0L}^2 = -e^{-2ip} \quad (4.60)$$

up to two loops. We see that the two loop correction vanishes. It would be interesting to see at what loop order we get the first deviation from this result.

Finally, we notice that there are no poles associated with boundary bound states in this matrix. This confirms, at weak coupling, our assumption that there are no boundary degrees of freedom in this theory.

4.2.1 One loop test for the $SU(1|2)^2$ reflection matrix

In this section we will compare the reflection amplitudes of Y , \overline{Y} and W (\overline{W} should be the same as W) off a boundary that consists of a $Y = 0$ giant graviton brane. These calculations were performed at one loop in [23], where they have an expression for the one loop boundary hamiltonian in the $S0(6)$ sector. In our notation⁹ the results they obtain

⁹Among other things they define the origin of the chain at site 1 instead of site 0. This introduces some phases.

for scattering off a boundary (a $\det(Y)$ boundary) on the left are

$$R_W = e^{-ip} = R_{\overline{W}} \quad (4.61)$$

$$R_Y = -1 \quad (4.62)$$

$$R_{\bar{Y}} = -e^{-2ip} \quad (4.63)$$

Notice the quotients $\frac{R_W}{R_Y} = -e^{-ip}$ and $\frac{R_{\bar{Y}}}{R_Y} = e^{-2ip}$ are the ones predicted by our exact matrix (3.24), recalling the expressions (3.13) for the impurities. Also, the overall factors are the same as the ones calculated at two loops in this section.

4.3 The $SU(2|2)$ spectrum and reflection matrix off a $\det(Z)$ boundary

Let us now go through a similar calculation for the $SU(2|2)$ reflection matrix, which corresponds to the case that we have an open chain ending on a $\det(Z)$ operator. In this case the ground state is non trivial. As we argued before, the letter placed next to the determinant, $\det(Z)$, cannot be a Z . This means that, at the very least, one field gets trapped in between the vacuum described by a chain of Z s and the D-brane. Our simplest guess for this operator is $\mathcal{O}_Z(Y, \dots)$, (2.4), where the dots represents the other boundary which we are not discussing now. Direct computation shows that this is an eigenstate with energy

$$H\mathcal{O}_Z(Y, \dots) = (2g^2 - 2g^4) \mathcal{O}_Z(Y, \dots) \quad (4.64)$$

This energy is the contribution from one boundary. In the case of the full chain, we have a second impurity at the other end and we have to add the corresponding energy. This energy agrees precisely with the weak coupling expansion of the exact formula (3.39),

$$\epsilon_B = \sqrt{1 + 4g^2} \sim 1 + 2g^2 - 2g^4 \quad (4.65)$$

This computation tests the boundary term in the Hamiltonian (4.54).

Once again, scattering states have the same energy as in the bulk, so the total energy is

$$H\mathcal{O}_Z(Y, Y_p, \dots) = \left[(2g^2 - 2g^4) + (8g^2 \sin^2 \frac{p}{2} - 32g^4 \sin^4 \frac{p}{2}) \right] \mathcal{O}_Z(Y, Y_p, \dots) \quad (4.66)$$

In appendix B we construct explicitly the wavefunction up to two loops, check this expression for the energy, and compute the reflection amplitude to two loops. We find

$$R' = -\frac{1 - 2e^{ip}}{1 - 2e^{-ip}} + 2g^2 \frac{e^{-ip}(e^{ip} - 1)^3(e^{ip} + 1)(1 - 4e^{ip} + e^{i2p})}{(e^{ip} - 2)^2} \quad (4.67)$$

This fixes the overall phase \mathcal{R}_{0L} in (3.46) at two loops for weak coupling. We would like to write this expression as a function of x^\pm, x_B such that we can make a guess that might be correct to a few higher orders as in [48]. Moreover, writing the expression this way allows for the identification of poles in the reflection matrix in a straightforward way. Notice that the coefficient A in the matrix \mathcal{R} (3.46) has the right limit at 1 loop but disagrees with (4.67) at two loops. We propose an expression that coincides with (4.67) up to two loops.

$$R' = - \frac{(x^+ + x_B) \left(x^+ + \frac{1}{x_B}\right) (x^- + x_B) \left(x^- + \frac{1}{x_B}\right)}{(x^- - x_B) \left(x^- - \frac{1}{x_B}\right) (x^+ - x_B) \left(x^+ - \frac{1}{x_B}\right)} \quad (4.68)$$

In checking this it is useful to remember the weak coupling expansions

$$x_B = \frac{i}{2g} \left(1 + \sqrt{1 + 4g^2}\right) \sim \frac{i}{g} + ig + \dots \quad (4.69)$$

$$x^\pm = e^{\pm i\frac{p}{2}} \frac{\left(1 + \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}\right)}{4g \sin \frac{p}{2}} \sim e^{\pm i\frac{p}{2}} \left(\frac{1}{g 2 \sin \frac{p}{2}} + 2g \sin \frac{p}{2} + \dots\right) \quad (4.70)$$

This expression for R' presents four simple poles. The pole at $x^- = x_B$ is responsible for the singularities of the weak coupling expansion (4.67). This is the pole that is already visible at one loop. This pole gives rise to a bound state in the s-channel and corresponds to the BPS boundary bound states that we discussed in section 3.2.1. We do not know if all the other poles of (4.67) survive when we add higher order corrections. It should be possible to perform an analysis similar to the one in [46], to determine the presence or absence of the other poles.

We can now also read off the two loop value of \mathcal{R}_{0L} in (3.46)

$$\mathcal{R}_{0L}^2 = \frac{R'}{A^2} = - \left(\frac{x^-}{x^+}\right)^2 \frac{(x^- - x_B)}{(x^+ - x_B)} \frac{\left(x^+ + \frac{1}{x_B}\right)}{\left(x^+ - \frac{1}{x_B}\right)} \frac{(x^- + x_B)}{(x^+ + x_B)} \frac{\left(x^- + \frac{1}{x_B}\right)}{\left(x^- - \frac{1}{x_B}\right)} \quad (4.71)$$

4.3.1 One loop test for the $SU(2|2)^2$ reflection matrix

We compare our exact results for the reflection matrix, (3.46), with the weak coupling results, as we did for the $SU(1|2)^2$ case. Unlike the previous case, this calculation is not available in the literature. We will need to compute the scattering process of a W approaching a $Z = 0$ brane with a Y degree of freedom. At one loop the fermions do not play a role and we can consider the $SU(3)$ sector to be closed. (This can be seen from the

expression of C in the exact solution, which is $\mathcal{O}(g)$ while A and B are of order unity). Therefore, our process is

$$|Y_B W_p\rangle \rightarrow R'_W |Y_B W_{-p}\rangle + R'_Y |W_B Y_{-p}\rangle \quad (4.72)$$

The Hamiltonian at one loop for the $SU(3)$ sector can be obtained by restricting the $SO(6)$ result in [23]. In our notation this is

$$H = 2g^2 \left(\sum_{i=0}^{\infty} (I - P_{i,i+1}) + P_{0,1} q_1^Z \right) \quad (4.73)$$

This means that when there is Z in the first (1) site it is the same as in the $SU(2)$ subsector, but the permutation operator does contribute when Y and W occupy the 0 and 1 site as opposed to the $SU(2)$ case. The reason for this is obvious: both Y and W can appear next to the determinant of Z s. We use the following trial eigenstate:

$$\Psi = \sum_{x=1}^{\infty} (e^{ipx} + R'_W e^{-ipx}) |Y_B W_x\rangle + R'_Y e^{-ipx} |W_B Y_x\rangle \quad (4.74)$$

where $|X_B^1 X_x^2\rangle$ is a state with an X^1 at the boundary (the site labelled by zero) and an X^2 at position x . In the bulk ($x > 1$) the eigenvalue equation yields the necessary value of the energy for both W and Y states.

$$E = 2g^2 (1 + 2 - e^{ip} - e^{-ip}) \quad (4.75)$$

Let us see what happens for the first site

$$E \begin{pmatrix} \psi_W(1) \\ \psi_Y(1) \end{pmatrix} = \begin{pmatrix} 2\psi_W(1) - \psi_W(2) - \psi_Y(1) \\ 2\psi_Y(1) - \psi_Y(2) - \psi_W(1) \end{pmatrix} \quad (4.76)$$

where $\psi_W = e^{ipx} + R'_W e^{-ipx}$ and $\psi_Y = R'_Y e^{-ipx}$. Using the bulk equations we get

$$\begin{pmatrix} \psi_W(1) \\ \psi_Y(1) \end{pmatrix} = \begin{pmatrix} \psi_W(0) - \psi_Y(1) \\ \psi_Y(0) - \psi_W(1) \end{pmatrix} \quad (4.77)$$

Plugging the ansatz for the wave function we get

$$R'_W = \frac{e^{2ip} - e^{ip} + 1}{e^{ip} - 2} \quad (4.78)$$

$$R'_Y = \frac{e^{2ip} - 1}{e^{ip} - 2} \quad (4.79)$$

These values satisfy $|R'_Y|^2 + |R'_W|^2 = 1$ as they should to comply with unitarity. Now we can compare the quotients $\frac{R'_W}{R'}$ and $\frac{R'_Y}{R'}$ with the expected values from the exact calculations, (3.46). Here R' is the value encountered in the $SU(2)$ sector at one loop (4.67). Namely,

$$R' = \frac{2e^{ip} - 1}{1 - 2e^{-ip}} \quad (4.80)$$

The resulting quotients are:

$$\frac{R'_W}{R'} = \frac{e^{ip} + e^{-ip} - 1}{2e^{ip} - 1}, \quad \frac{R'_Y}{R'} = \frac{e^{ip} - e^{-ip}}{2e^{ip} - 1} \quad (4.81)$$

From the exact result (3.46) we have

$$\frac{R'_W}{R'} = \frac{1}{2} \left(1 + \frac{B}{A} \right), \quad \frac{R'_Y}{R'} = \frac{1}{2} \left(1 - \frac{B}{A} \right) \quad (4.82)$$

Expanding A , B , using the first terms in (4.69)(4.70), we checked that these equations are true. This is a nontrivial one loop check for the bosonic subsector of the reflection matrix. A very easy check is that $R'_Y + R'_W = R'$.

5 Results at strong coupling

In this section, we discuss results obtained in the strong coupling regime from string theory. As long as one is interested in the leading terms in g , it is possible to calculate scattering amplitudes by calculating time delays in classical sine Gordon theory [34]. We make use of this possibility to calculate the overall phase of the reflection matrix at strong coupling for both the $Z = 0$ and $Y = 0$ giant graviton branes. To be more precise, at strong coupling there are three regimes, depending on how we scale the momentum. We can keep the momentum fixed and then compute as we mentioned above; this is the giant magnon regime. We could also scale the momentum as $p \sim 1/g$ and this corresponds to the near plane wave limit. Finally we can set $p \sim 1/\sqrt{g}$, see [49]. For the case of bulk scattering it is possible to write a formula which captures the leading order result both in the plane wave and giant magnon regimes [7]. Here we will focus on the giant magnon region. As we briefly discussed in section 3.1.3, the result in the plane wave region is trivial. Some results in the near plane wave region were obtained in [21].

5.1 Boundary conditions in the sine Gordon theory

According to the work of Pohlmeyer [50] it is possible to map the problem of a string propagating on $\mathbb{R} \times S^2$ into the classical sine Gordon model, see also [51]. This connection

was used in [34] to calculate the strong coupling limit of the bulk scattering phase of string theory on $AdS_5 \times S^5$. We will do the same here.

We use string worldsheet coordinates in which $t = 1$. Then, the sine Gordon field, $\phi(x, t)$, is related to the unit vector $\boldsymbol{\eta}$ describing the S^2 as

$$\cos 2\phi = \dot{\boldsymbol{\eta}}^2 - \boldsymbol{\eta}'^2 \quad (5.83)$$

where

$$\boldsymbol{\eta}^2 = 1, \quad \dot{\boldsymbol{\eta}}^2 + \boldsymbol{\eta}'^2 = 1, \quad \dot{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}' = 0 \quad (5.84)$$

We can consider simple cases leading to different boundary conditions for the sine Gordon theory.

1. Scattering off a $Z = 0$ giant graviton brane
2. Scattering off a $Y = 0$ giant graviton brane where we chose the S^2 within brane, e.g. the S^2 given by $|Z|^2 + (\Phi_1)^2 = 1$
3. Scattering off a $Y = 0$ giant graviton brane where we chose the S^2 transverse to the brane, e.g. the S^2 given by $|Z|^2 + (\Phi_3)^2 = 1$

Recall that $Z = \Phi^5 + i\Phi^6$, $Y = \Phi^3 + i\Phi^4$.

In the first case the boundary is fixed at the center of the Z plane. This means that the S^2 boundary condition is $\dot{\boldsymbol{\eta}}|_{\text{Boundary}} = 0$. Therefore, using equations (5.83) and (5.84), we find the Dirichlet boundary condition $\phi|_{\text{Boundary}} = \frac{\pi}{2}$. This type of boundary conditions were discussed for the classical sine Gordon theory in [52] and the time delay was calculated. Note that $\phi = \frac{\pi}{2}$ corresponds to the maximum of the sine Gordon potential. This implies that the field has to move from the maximum to the minimum and this leads to some energy that is localized near the boundary. This corresponds to the boundary degree of freedom, or boundary impurity, that we discussed above.

The second case represents a string that is entirely contained inside the D-brane that it is attached to. Therefore, the string end point (the one ending on the D-brane) can move freely on the S^2 , thus $\boldsymbol{\eta}' = 0$ and this leads to another Dirichlet boundary condition

for the sine Gordon field $\phi|_{\text{Boundary}} = 0$. In this case the field is at the minimum of the potential and we have nothing localized at the boundary.

Finally, in the third case the endpoint of the string, which has to lie both on the D-brane and inside the S^2 , has to be on the rim of the disk $|z| = 1$, which is the only region common to both. One can then show that this leads to $\phi'|_{\text{boundary}} = 0$.

In this fashion, we see how different physical configurations in $AdS_5 \times S^5$ lead to different boundary problems for the sine Gordon theory. Interestingly enough, all the boundary conditions that were discussed belong to the special class that make the boundary field theory integrable [13]. Incidentally, the string theory setup we are studying was shown to be integrable at large g in [28]. It would be interesting to see if other integrable boundary conditions in the sine Gordon model map to other configurations in the string theory.

We should mention that this description that uses the sine Gordon theory is only an approximation (valid in the classical limit). It is not capturing the fact that there are collective coordinates characterizing the magnon. These arise because the magnon has an S^3 worth of possible orientations inside the S^5 . (In addition, we have fermion zero modes [53].) As we quantize these coordinates we get all the BPS bound states with various values of the angular momentum charge n [37, 38]. In particular, the fundamental impurities, such as the fields Y , X , etc, have wavefunctions that are spread over this S^3 . Thus, when we talked about solutions that were localized within a given S^2 , we were making an approximation where we neglected this motion. One could get a better approximation by considering the solutions in [54], which can be used to describe the classical limit of the scattering of BPS bound states [37] with angular momentum $n \sim \mathcal{O}(g)$ from the boundary. In the case of the $Z = 0$ brane, where we have a boundary impurity, we construct the solution as follows. Consider a soliton of the bulk theory with momentum $p = \pi$ that is at rest at the origin. This is a solution that obeys the boundary conditions of the boundary theory. Its energy is simply half of the energy of the original soliton. We can similarly consider the generalizations with angular momentum discussed in [38, 54]. In that case both the angular momentum and energy are half of what they were in the bulk. However, in the boundary case, we want to quantize the angular momentum so that it is an integer after dividing by half. Thus we get a formula for the energies that has the form

$$\epsilon_B = \frac{1}{2} \sqrt{(2n)^2 + 16g^2} = \sqrt{n^2 + 4g^2} \quad (5.85)$$

where n is an integer. This is in agreement with the exact results (3.48).

5.2 Time delays and scattering phases

Let us consider first the case where we have a $Y = 0$ giant graviton brane. It is convenient to think about the problem by using a “method of images” where the incoming soliton scatters an antisoliton or a soliton coming from the other side of the boundary, depending on the boundary conditions. From our experience with the sine Gordon model and the bulk calculations in [34], we know the result will be independent of whether the image state is a soliton or an antisoliton. Therefore, we don’t need to specify this in our calculations.

When we translate between the sine-Gordon results and the results computed in the conventions that are more natural at weak coupling we need to be careful about the fact that these two different conventions differ in the definition of the spatial coordinate. This was explained in more detail in [34, 55, 56]. In fact, we can work in conventions that coincide with the gauge theory conventions and notice that the classical boundary scattering amplitude has a simple relation to the bulk scattering amplitude once we note that the boundary scattering amplitude can be computed by the “method of images”. Let us consider the case where we scatter from a right boundary¹⁰.

For a $Y = 0$ brane, we have two solitons, one with momentum $p_1 = p$ and another with momentum $p_2 = -p$. The bulk scattering phase is related to the time delays

$$\Delta T_{12} = \frac{dp_1}{d\epsilon_1} \partial_{p_1} \delta(p_1, p_2) , \quad \Delta T_{21} = \frac{dp_2}{d\epsilon_2} \partial_{p_2} \delta(p_1, p_2) \quad (5.86)$$

where $\delta(p_1, p_2)$ is the bulk scattering phase computed in [34]

$$\delta(p_1, p_2) = -4g \left(\cos \frac{p_1}{2} - \cos \frac{p_2}{2} \right) \log \left[\frac{\sin^2 \frac{p_1 - p_2}{4}}{\sin^2 \frac{p_1 + p_2}{4}} \right] \quad (5.87)$$

where $\text{sign}(\sin p_i) > 0$. For $p_2 = -p < 0$ we should set $p_2 = 2\pi - p$ in this formula, and this is what we will always mean by $-p$. In the case that $p = p_1 = -p_2$ we find that the two time delays are equal to each other and to the time delay for scattering from the boundary $\Delta T_{12} = \Delta T_{21} = \Delta T_B(p)$. Thus we conclude that the classical (right) boundary scattering phase, $\mathcal{R}_R = e^{i\delta_{B,R}}$, is the solution to

$$\begin{aligned} \frac{dp}{d\epsilon} \partial_p \delta_{B,R}(p) &= \Delta T_B(p) = \frac{1}{2} (\Delta T_{12} + \Delta T_{21}) = \\ &= \frac{1}{2} \left(\frac{dp_1}{d\epsilon_1} \partial_{p_1} \delta(p_1, p_2) + \frac{dp_2}{d\epsilon_2} \partial_{p_2} \delta(p_1, p_2) \right) \Big|_{p_1 = -p_2 = p} \end{aligned} \quad (5.88)$$

¹⁰We can obtain the result for left boundaries by a parity transformation $\mathcal{R}_L(p) = \mathcal{R}_R(-p)$.

A solution to this equation is then

$$\delta_{B,R}(p) = \frac{1}{2}\delta(p, -p) = -8g \cos \frac{p}{2} \log \cos \frac{p}{2} \quad (5.89)$$

where δ is (5.87). This describes right-boundary scattering. Note that we get the same answer regardless of the state of the impurity, since the matrix structure of the reflection matrix (3.23) is subleading at large g . This also means that this is an actual calculation of the overall phase factor \mathcal{R}_{0R}^2 at strong coupling and to leading order.

We can check that this result obeys the classical limit of the crossing equation (3.31)

$$\delta_{B,R}(p) + \delta_{B,R}(\bar{p}) = -\delta(p, -\bar{p}) + \mathcal{O}(1) \quad (5.90)$$

where the $\mathcal{O}(1)$ terms are order one in the $1/g$ expansion. Notice that in order to get the results for \bar{p} , we should set $p \rightarrow -p$ in (5.87) (5.89) and, as we mentioned before, to get the results for $-p$ we should set $p \rightarrow 2\pi - p$ in (5.87) (5.89).

This result is valid in the giant magnon regime. We remind the reader that reflection becomes trivial in the plane wave region, as magnons become noninteracting. In that case, we get Dirichlet boundary conditions for the fields Y , \bar{Y} and Neumann for W , \bar{W} . This implies that $\mathcal{R}_{0R}^2 = -1$ in the plane wave regime.

In a similar way we can compute the classical limit of the boundary scattering for the $Z = 0$ brane. In this case we have a boundary impurity. Using the “method of images” we can represent the boundary impurity as a third soliton, with momentum $p = \pi$ that is sitting at the boundary. This type of solutions was obtained explicitly for the sine Gordon model in [52]. In order to compute right boundary scattering we consider a bulk configuration with three solitons with $p_1 = p$, $p_2 = -p$ and $p_3 = \pi$. Then the time delay is

$$\Delta T(p) = \Delta T_{12} + \Delta T_{13} = \frac{1}{2}(\Delta T_{12} + \Delta T_{21}) + \Delta T_{13} \quad (5.91)$$

Writing this as in (5.88) we find the large coupling expression for the phase in (3.46), $\mathcal{R}_{0,R}^2 = e^{i\delta_{B,R}^Z}$,

$$\delta_{B,R}^Z(p) = \frac{1}{2}\delta(p, -p) + \delta(p, \pi) = -4g \cos \frac{p}{2} \log \left[\cos^2 \frac{p}{2} \frac{(1 - \sin \frac{p}{2})}{(1 + \sin \frac{p}{2})} \right] \quad (5.92)$$

where

$$\delta(p, \pi) = -4g \cos \frac{p}{2} \log \left[\frac{1 - \sin \frac{p}{2}}{1 + \sin \frac{p}{2}} \right] \quad (5.93)$$

The classical limit of the crossing symmetry equation is expected to be similar and it would still be obeyed since (5.93) is odd under $p \rightarrow -p$ (which is what we should do to cross $p \rightarrow \bar{p}$).

6 Conclusions and discussion

6.1 Summary of results

In this article we considered some D-brane configurations in $AdS_5 \times S^5$ and considered the worldsheet theory of an open string ending on the D-brane. We focused on the D-branes that correspond to maximal giant gravitons. In the dual field theory, these D-branes correspond to determinant operators of the form $\det(Y)$, $\det(Z)$, where Y , Z are two complex combinations of the scalar fields in $\mathcal{N} = 4$ super Yang Mills. We considered an open string attached to this operator with a large value of J , where J is one of the generators of $SO(6)$. In the dual field theory this corresponds to attaching a long string of Z s to the determinant operator. This can be viewed as a spin chain defined on an interval. We then considered impurities propagating on this chain of Z s. The symmetries of the problem determine completely the single impurity reflection matrix up to an overall phase. These reflection matrices are asymptotic, as in the bulk [10]. Namely, we need to go far away from the boundary to measure it. Thus, the strict mathematical definition of the reflection matrix requires $J = \infty$.

We considered two cases. First the case where the determinant operator was $\det(Y)$. In this case the boundary breaks the bulk symmetry group to an $SU(1|2)^2$ subgroup. Yet, this symmetry is powerful enough to determine the matrix structure of the reflection matrix. In fact, in a natural basis, the reflection matrix is diagonal.

We then considered the case where we have a $\det(Z)$ operator. In this case an impurity gets trapped between the string of Z s describing the open string ground state and the determinant operator. This impurity acts as a boundary degree of freedom. This problem respects the full extended $SU(2|2)^2$ symmetry that we have on the bulk of a chain of Z s, or the bulk of the string in light cone gauge [8]. The boundary impurity transforms in the fundamental representation of the extended $SU(2|2)^2$ algebra and has a (complex) central charge with fixed modulus and a phase that is determined by the momenta of the other particles. This is very similar to the structure we have in the bulk of the string. The algebra determines the energy of the boundary impurity. In this case, the reflection

matrix acts on the boundary degree of freedom. The resulting matrix is rather similar to the one describing the bulk scattering of two impurities [1]. Also, the bulk particle can form BPS bound states with the boundary degrees of freedom. Thus, the spectrum of boundary degrees of freedom includes an index n which characterizes the total number of impurities forming the bound state.

Both of reflection matrices obey the boundary Yang Baxter equation, which is a requisite for integrability. In the first case, we derived explicitly the form of the crossing equation by considering the scattering of a particle/hole pair and demanding that the corresponding reflection amplitude is trivial. This derivation could be extended to the second case in a straightforward way.

We then performed computations in the weak coupling regime. Here we checked the integrability of the system up to two loops. We resolved the problems raised in [31] by noticing that there is an extra boundary contribution to the spin chain Hamiltonian. The results we obtain at two loops are consistent with integrability, in the sense that the asymptotic Bethe ansatz for two particles works properly. Nevertheless, we have not proven the full integrability of the system at two loops. We also computed the undetermined phase factor in the reflection matrix up to two loops in the weak coupling expansion. In addition, we checked that the matrix structure obtained by the symmetry arguments was consistent with the explicit weak coupling results.

We also computed the strong coupling limit of the reflection phase. At strong coupling there are two perturbative regimes, the near plane wave regime and the giant magnon regime, depending on the momentum of the impurity. We computed the leading order result for the scattering amplitude in the giant magnon regime. The computation can be carried out in a simple way by using a “method of images”, where we view the problem with a boundary in terms of a problem on the full line with the proper symmetry under reflection¹¹. This gives the boundary scattering phase in terms of the bulk scattering phase.

Note that our computations of the matrix structure of the reflection matrix are valid also for other systems where we have $SU(2|2)$ symmetry. One such system is the plane wave matrix model [9], where one can study configurations analogous to the ones considered here, even though this particular system appears not to be integrable [57].

¹¹ This method is useful for the classical theory but it is not appropriate for the full quantum theory.

6.2 Problems for the future

We would now like to point out to some open directions that seem worth exploring further.

The most obvious open problem is to find the overall phase factor by solving the crossing equation, as was done for the bulk in [3].

Once we know the phase for the two cases, then, one can check that we get a consistent result by starting with the $\det(Y)$ brane (or $Y = 0$ giant graviton brane) and fill in the vacuum by adding Y impurities until all we have are two Z s that get trapped at the ends. This should correctly reproduce the energy of the ground state for an open string on a $\det(Z)$ brane (or $Z = 0$ giant graviton brane) containing one impurity at each end. This gives a consistency check. Alternatively, if we assume it is true, this could give us a method for computing the reflection phase for one case once we know it for the other case.

Once one has found the overall phase, then one can write Bethe equations that determine the energy of the system. These equations will describe only the large J limit of the system. To go to the limit of small J one will have to use some more clever methods, which hopefully rely only on the reflection matrix that we are considering here. Some finite J corrections were computed in [58], for the closed string case.

It seems possible to study other D-branes in the bulk. For example, D-branes that are associated to adding flavors to the theory or D-branes that correspond to adding operators with various codimensions in the boundary theory. It seems that many of these cases could be solved by the techniques in this paper, since they appear to have enough symmetry to completely constrain the reflection matrix.

Another interesting case to analyze is the situation where we have local operators on a half BPS Wilson line [17]. When we consider operators with large J we get an open spin chain. The boundary conditions seem to preserve a diagonal $SU(2|2)$ subgroup. This is likely to be enough to fix the reflection matrix completely.

It seems that one could extend our computations to the case of non-maximal giants, which was considered in [24]. We again preserve the full extended $SU(2|2)^2$ symmetry, but the boundary impurity has a central charge whose absolute value also depends on its phase, see figure 11. If we are dealing with a semi-infinite chain, then we could compute the matrix structure of the reflection amplitude with the methods of this paper. That computation does not rely on integrability. It remains to be seen whether the system is integrable or not in this case.

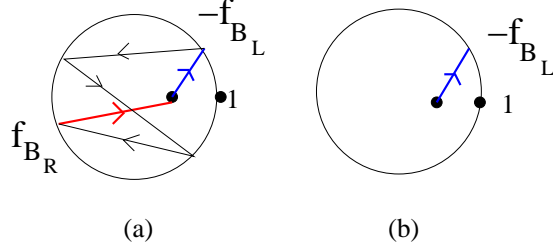


Figure 11: (a) Open string configuration with very large J containing three separated magnons which ends on a non-maximal giant graviton. In (b) we isolated one of the boundary impurities. Note that the length of the boundary impurity line depends on the point along the circle where it ends.

Acknowledgments

We would like to thank A. Agarwal and N. Beisert for discussions. This work was supported in part by DOE grant #DE-FG02-90ER40542.

A Appendix: integrability at two loops

It was pointed out in [31] that the Bethe ansatz seems to fail at two loops for the problem just studied. We will now show that the problems raised disappear once we consider the correct Hamiltonian (4.54). In particular, the problem was found when one tried to construct a two particle state using the original scattering data.

We will consider a wave function of the form $\Psi(x, y) = \Psi_0(x, y) + g^{2|x-y|}\Upsilon(x, y)$ where we will only be concerned with corrections of order g^2 to the standard Bethe ansatz wave function $\Psi_0(x, y)$. This is the asymptotic Bethe ansatz discussed in [10]. Our state is

$$\mathcal{O}_Y(Y_{p_1}Y_{p_2}) = \sum_{0 < x < y}^{\infty} \Psi(x, y) \mathcal{O}_Y(Y_x Y_y) \quad (\text{A.94})$$

The equations we have to satisfy in the bulk are

$$\begin{aligned}
E\Psi(x, y) = & (2g^2 - 8g^4) (4\Psi(x, y) - \Psi(x-1, y) - \Psi(x+1, y) - \Psi(x, y-1) - \Psi(x, y+1)) \\
& + 2g^4 (4\Psi(x, y) - \Psi(x-2, y) - \\
& \Psi(x+2, y) - \Psi(x, y-2) - \Psi(x, y+2)) \quad \text{for } 2 < x < y-2
\end{aligned} \tag{A.95}$$

$$\begin{aligned}
E\Psi(x, x+2) = & (2g^2 - 8g^4) (4\Psi(x, x+2) - \Psi(x-1, x+2) - \Psi(x+1, x+2) \\
& - \Psi(x, x+1) - \Psi(x, x+3)) + 2g^4 (2\Psi(x, x+2) - \Psi(x-2, x+2) \\
& - \Psi(x, x+4)) \quad \text{for } 2 < x
\end{aligned} \tag{A.96}$$

$$\begin{aligned}
E\Psi(x, x+1) = & (2g^2 - 8g^4) (2\Psi(x, x+1) - \Psi(x-1, x+1) - \Psi(x, x+2)) \\
& + 2g^4 (4\Psi(x, x+1) - \Psi(x-2, x+1) - \\
& \Psi(x+1, x+2) - \Psi(x-1, x) - \Psi(x, x+3)) \quad \text{for } 2 < x
\end{aligned} \tag{A.97}$$

where E is the sum of the one particle energies. These equations specify $\Upsilon(x, y)$ completely, as well as the bulk scattering matrix [10] [6]. In order to obtain information about the reflection matrix we need to check the eigenvalue equation for sites close to the boundary. If we pick sites of the form $(2, x)$ our equations are:

$$\begin{aligned}
E\Psi(2, x) = & (2g^2 - 8g^4) (4\Psi(2, x) - \Psi(1, x) - \Psi(3, x) - \Psi(2, x-1) - \Psi(2, x+1)) \\
& + 2g^4 (3\Psi(2, x) - \Psi(4, x) - \\
& \Psi(2, x-2) - \Psi(2, x+2)) + 2g^4 \Psi(2, x) \quad \text{for } 4 < x
\end{aligned} \tag{A.98}$$

$$\begin{aligned}
E\Psi(2, 4) = & (2g^2 - 8g^4) (4\Psi(2, 4) - \Psi(1, 4) - \Psi(3, 4) - \Psi(2, 3) - \Psi(2, 5)) \\
& + 2g^4 (\Psi(2, 4) - \Psi(2, 6)) + 2g^4 \Psi(2, 4)
\end{aligned} \tag{A.99}$$

$$\begin{aligned}
E\Psi(2, 3) = & (2g^2 - 8g^4) (2\Psi(2, 3) - \Psi(1, 3) - \Psi(2, 4)) \\
& + 2g^4 (3\Psi(2, 3) - \Psi(2, 5) - \Psi(3, 4) - \Psi(1, 2)) + 2g^4 \Psi(2, 3)
\end{aligned} \tag{A.100}$$

If we use the original equations, these just imply $\Psi(0, x) = \Psi_0(0, x) = 0$ for $x > 2$. These are the analogous equations to $\Psi(0) = 0$ in the single particle case and determine the one particle reflection matrix to be consistent with the Bethe ansatz.

We still have to consider the sites $(1, x)$. These can't introduce any more constraints,

as our function is already fully determined. The resulting equations are:

$$\begin{aligned}
E\Psi(1, x) = & (2g^2 - 8g^4)(3\Psi(1, x) - \Psi(2, x) - \Psi(1, x+1) - \Psi(1, x-1)) \\
& + 2g^4(3\Psi(1, x) - \Psi(3, x) - \Psi(1, x-2) - \Psi(1, x+2)) + \\
& (2g^2 - 4g^4)\Psi(1, x) \quad \text{for } 3 < x
\end{aligned} \tag{A.101}$$

$$\begin{aligned}
E\Psi(1, 3) = & (2g^2 - 8g^4)(3\Psi(1, 3) - \Psi(2, 3) - \Psi(1, 4) - \Psi(1, 2)) \\
& + 2g^4(\Psi(1, 3) - \Psi(1, 5)) + (2g^2 - 4g^4)\Psi(1, 3)
\end{aligned} \tag{A.102}$$

$$\begin{aligned}
E\Psi(1, 2) = & (2g^2 - 8g^4)(\Psi(1, 2) - \Psi(1, 3)) \\
& + 2g^4(2\Psi(1, 2) - \Psi(1, 4) - \Psi(2, 3)) + (2g^2 - 2g^4)\Psi(1, 2)
\end{aligned} \tag{A.103}$$

Making use of the bulk equations the first of these expressions yields $\Psi(-1, x) + \Psi(1, x) = \Psi_0(-1, x) + \Psi_0(1, x) = 0$ for $x > 3$. These are the analog of $\Psi(1) + \Psi(-1) = 0$ and impose no further constraints, as our wave function satisfies this identity. The second equation gives the same result for $x = 3$. The last of these equations is the one that presented a conflict in [31]. In our case this equation can be written (to order g^4) as

$$2g^4(\Psi_0(1, 2) + \Psi_0(-1, 2)) + (2g^2 - 8g^4)\Psi_0(0, 2) + 2g^4\Psi_0(0, 1) = 0 \tag{A.104}$$

This is satisfied by our Bethe ansatz as $\Psi_0(1, 2) + \Psi_0(-1, 2) = 0$, $\Psi_0(0, 2) = 0$ and $\Psi_0(0, 1) = 0$. This shows that the two particle problem can be solved by the asymptotic Bethe ansatz technique, suggesting integrability.

B Appendix: computation of the $SU(2|2)$ reflection matrix at two loops

The wave function for a one particle state scattering of the boundary should satisfy:

$$\begin{aligned}
E\Psi(x) = & (2g^2 - 8g^4)(2\Psi(x) - \Psi(x+1) - \Psi(x-1)) \\
& + 2g^4(2\Psi(x) - \Psi(x+2) - \Psi(x-2)) \\
& + (2g^2 - 2g^4)\Psi(x) \quad \text{for } x > 2
\end{aligned} \tag{B.105}$$

for the trial wave function $\Psi(x) = \Psi_0(x) + g^2\delta_{x,1}\Upsilon$. The g^2 correction is just an exponential tail attached to the boundary that accounts for the interactions at two loops. Further corrections are higher order in g^2 . $\Psi_0(x)$ is just the reflecting wave solution $\Psi_0(x) =$

$e^{ipx} + R'e^{-ipx}$, where R' has, in principle, g^2 corrections to the 1 loop result. From this expression we check that the energy of this state is indeed (4.66).

The equation that determines Υ comes from the coefficient of the Schrodinger equation for site 2. Namely

$$\begin{aligned} E\Psi(2) &= (2g^2 - 8g^4)(2\Psi(2) - \Psi(3) - \Psi_0(1)) - 2g^4\Upsilon \\ &\quad + 2g^4(\Psi(2) - \Psi(4)) + (2g^2 - 4g^4)\Psi(2) \end{aligned} \quad (\text{B.106})$$

Using the bulk equation (B.105) we get

$$\Upsilon = \Psi(0) - 2\Psi(2) \quad (\text{B.107})$$

The equation at site 1 determines the reflection amplitude. This is

$$\begin{aligned} E\Psi_0(1) + 2g^4(3 - e^{ip} - e^{-ip})\Upsilon &= \\ (2g^2 - 8g^4)(\Psi_0(1) - \Psi(2)) + 2g^4(\Psi_0(1) - \Psi(3)) + \\ 2g^4\Upsilon + 2g^4\Psi_0(1) \end{aligned} \quad (\text{B.108})$$

where $2g^2(3 - e^{ip} - e^{-ip})$ is the one loop energy extracted from (4.66). Using the bulk equation we get

$$\begin{aligned} 2g^4(2 - e^{ip} - e^{-ip})\Upsilon &= \\ (10g^4 - 4g^2)\Psi_0(1) + (2g^2 - 8g^4)\Psi_0(0) + 2g^4\Psi_0(-1) \end{aligned} \quad (\text{B.109})$$

Plugging in for Υ and the wave function we get

$$\begin{aligned} &2g^4(2 - e^{ip} - e^{-ip}) - 4g^4(2e^{i2p} - e^{i3p} - e^{ip}) \\ &-(10g^4 - 4g^2)e^{ip} - (2g^2 - 8g^4) - 2g^4e^{-ip} = \\ &-R'[2g^4(2 - e^{ip} - e^{-ip}) - 4g^4(2e^{-i2p} - e^{-ip} - e^{-i3p}) \\ &-(10g^4 - 4g^2)e^{-ip} - (2g^2 - 8g^4) - 2g^4e^{ip}] \end{aligned} \quad (\text{B.110})$$

This in turn implies the weak coupling expansion

$$R' = -\frac{1 - 2e^{ip}}{1 - 2e^{-ip}} + 2g^2 \frac{e^{-ip}(e^{ip} - 1)^3(e^{ip} + 1)(1 - 4e^{ip} + e^{i2p})}{(e^{ip} - 2)^2} \quad (\text{B.111})$$

This is the result (4.67).

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